Periodic cyclic homology of smooth crossed product algebras

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Abstract

This paper is devoted to the study of smooth convolution algebras associated to Lie groups, appearing naturally in non-commutative geometry and representation theory. Our result establishes a stable isomorphism between the periodic cyclic homology of the smooth crossed product algebras associated to a real Lie group and to a maximal compact subgroup of its. It may be viewed as an homological analogue to the Dirac induction with coefficients for Lie groups. Our approach is built on earlier work of Nistor who established this isomorphism locally, *i.e.* around each conjugacy class of the group and without stabilization. Essential tools come from Kasparov's bivariant K-theory and its interpretation by Cuntz. We set up a refinement of the Dirac-dual Dirac method and of Morita equivalence which fit into our framework.

Contents

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1 Introduction

Let G be a real reductive Lie group and let K be a maximal compact subgroup. The contractibility of the homogeneous space G/K provides one of the central bridges between representation theory, index theory, and non-commutative geometry. A recurring principle is that, although G might be analytically complicated, many of its invariants can be compared with invariants attached to the compact group K. The present work develops such a comparison in periodic cyclic homology for smooth crossed product algebras.

The starting point is the Connes-Kasparov philosophy. For a connected real reductive group G, the Connes-Kasparov theorem provides an isomorphism up to a degree shift

$$K_\bullet(C_r^\star(K)) \xrightarrow{\sim} K_{\bullet+\dim(G/K)}(C_r^\star(G))$$

between the K-theoretical invariants of the associated reduced $C^\star$-algebras. This comparison is implemented by Dirac induction. The Dirac operator on the symmetric space G/K, together with its dual class, gives rise to mutually inverse elements in equivariant Kasparov theory. Representation-theoretically, this result is related to the correspondence between the unitary dual of K and the tempered dual of G, as developed in the work of Wassermann and later clarified through the Mackey analogy. Geometrically, the deformation of G to its Cartan motion group explains why the compact subgroup K should control the K-theory of G.

If the real Lie group G acts on a $C^\star$-algebra A, we can construct a reduced crossed product $A \rtimes_r G$ which can be thought as a reduced group $C^\star$-algebra with coefficients and encodes both representations of the group G and modules over the algebra A. Naturally, one might consider a version of the Connes-Kasparov isomorphism with coefficients. It is the purpose of the Baum-Connes conjecture with coefficients for Lie groups, which is still unproved.

Conjecture 1. *There exists a degree-shift isomorphism*

$$K_\bullet(A \rtimes_r K) \longrightarrow K_{\bullet+\dim(G/K)}(A \rtimes_r G). \tag{1}$$

The question considered here belongs to the homological side of this picture. If A is a locally convex algebra endowed with a smooth action of G, the smooth crossed product $A \rtimes G$ is a convolution algebra which both combines the algebraic structure of A and the dynamics of the action. Such algebras arise naturally in non-commutative geometry, and should be viewed as smooth counterparts of reduced crossed product $C^\star$-algebras. Whereas the latter are adapted to K-theoretical invariants, smooth crossed products are especially suitable for periodic cyclic homology.

Periodic cyclic homology is the appropriate homological invariant in this setting. For the algebra of smooth functions on a compact manifold, the Hochschild-Kostant-Rosenberg theorem identifies periodic cyclic homology with the even-odd graded De Rham cohomology. In this sense, periodic cyclic homology extends De Rham theory from manifolds to non-commutative algebras. It also shares a number of formal properties with K-theory, namely homotopy invariance, excision, Morita invariance, and compatibility with an algebraic Chern character. These analogies suggest that the Connes-Kasparov isomorphism should have a cyclic-homological counterpart.

For smooth crossed products, this expectation was explored by Nistor in the early 1990s. He showed that the periodic cyclic homology of $A \rtimes G$ and $A \rtimes K$ are modules over the algebra of smooth class functions on G, and isomorphic after localization at maximal ideals of this algebra.

Theorem (Nistor [Nis93])**.** *Let A be locally convex algebra endowed with a smooth action of G. For any maximal ideal $\mathfrak{m}$ of the smooth class functions algebra $\mathcal{C}^\infty(G)^{inv}$, there exists an isomorphism functorial in A*

$$HP_\bullet(A \rtimes K)_\mathfrak{m} \simeq HP_{\bullet+\dim(G/K)}(A \rtimes G)_\mathfrak{m}.$$

This result can be thought as a *local* homological-analogue of the Baum-Connes conjecture with coefficiants for Lie groups. It is already a strong indication that the geometry of G/K governs the cyclic theory of smooth crossed products, just as it governs the K-theory of reduced crossed products. Nevertheless, the passage from local statements to a global isomorphism is subtle. The local description involves centralizers of elements of G and their geometry does not vary smoothly enough across conjugacy classes to permit a direct gluing argument. A global construction requires a mechanism that avoids decomposing the group into local contributions and instead works directly with the geometry of G/K.

The purpose of this paper is to provide such a global construction after a natural stabilization. Our main result establishes that for any locally convex algebra A endowed with a smooth action of G, there exists an isomorphism

$$HP_\bullet(\mathcal{A} \rtimes K) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}(\mathcal{A} \rtimes G) \tag{2}$$

after stabilization $\mathcal{A} = A \otimes_\pi \mathcal{D}$ by a certain G-smooth Schatten ideal $\mathcal{D}$. This statement may be regarded as an homological analogue of Dirac induction with coefficients. Two ingredients organize the proof.

The first is a smooth version of Green's imprimitivity theorem. In the locally convex setting, usual Morita equivalence is replaced by a weaker notion, called quasi-Morita equivalence. The Cuntz-Quillen description of periodic cyclic homology using quasi-free extensions and X-complexes shows that quasi-Morita equivalence yields chain-homotopy equivalence in periodic cyclic homology. This paper develops a smooth-equivariant version of this equivalence according to [Cun98], compatible with periodic cyclic homology

$$A \sim_{G-\text{quasi-Morita}} B \;\Rightarrow\; \widehat{CC}(A \rtimes G) \xrightarrow{\sim} \widehat{CC}(B \rtimes G).$$

We show that, for any closed subgroup $H < G$, the locally convex G-algebras $\mathcal{C}_c^\infty(G, A) \rtimes H$ and $\mathcal{C}_c^\infty(G/H, A)$ are G-equivariantly quasi-Morita equivalent. Once applied to the closed subgroups $H = G$ and $H = K$, it yields to the following isomorphism.

Theorem. *For any locally convex algebra A endowed with a smooth action of G, there exists an isomorphism*

$$HP_\bullet(A \rtimes K) \xrightarrow{\sim} HP_\bullet(\mathcal{C}_c^\infty(G/K, A) \rtimes G). \tag{3}$$

The second ingredient is obtained by adapting Kasparov's Dirac-dual Dirac method to periodic cyclic homology. On the symmetric space G/K, Kasparov constructed a G-equivariant Dirac element α and a K-equivariant dual Dirac element β whose Kasparov products are inverse after restriction to the maximal compact subgroup K [Kas81]. This equivariant duality is the main argument behind the proof of the Connes-Kasparov theorem. To transport it to smooth crossed products, one needs an algebraic model of bivariant classes that is compatible with Fréchet algebras and with crossed products.
This paper provides such a model by developing an analogue of the Cuntz picture [Cun87] in the Fréchet framework. Classically, a KK-class is represented by a $\star$-homomorphism out of the Cuntz algebra qA, and the Kasparov product corresponds to composition after stabilization.

Here we introduce a Fréchet version $q_R A$ of the Cuntz algebra for fixed parameter $R > 0$, defined as a completion of the algebraic free product with respect to a family of submultiplicative norms. This construction is functorial, stable under group actions, and compatible with crossed products. Within this setting, the Dirac and dual Dirac elements can be represented by equivariant homomorphisms of Fréchet algebras and induce linear maps between the periodic cyclic complexes of the corresponding crossed products by the Cuntz-Quillen formalism

$$HP(\alpha \rtimes G) : HP_\bullet(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes G) \longrightarrow HP_{\bullet+\dim(G/K)}(\mathcal{A} \rtimes G)$$

$$HP(\beta \rtimes K) : HP_\bullet(\mathcal{A} \rtimes K) \longrightarrow HP_{\bullet+\dim(G/K)}(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes K)$$

where $\mathcal{A} = A \otimes_\pi \mathcal{D}$ denotes the stabilization of the algebra A with a G-smooth Schatten ideal $\mathcal{D}$. Indeed, the linear maps $HP(\alpha \rtimes K)$ and $HP(\beta \rtimes K)$ are not immediately inverse to each others on periodic cyclic homology and the stabilization algebra $\mathcal{D}$ is designed precisely to absorb this defect.

Finally, the crucial observation is that their K-equivariant duality suffices to produce a global isomorphism after passing to crossed products by the whole G. Indeed, the Nistor's local comparison theorem applied to the map $HP(\alpha \rtimes G)$ shows that it becomes an isomorphism on every stalk, namely a global isomorphism. In this way, the K-equivariant duality of Dirac and dual Dirac elements transported to periodic cyclic homology, yields the degree-shift isomorphism.

Theorem. *Let A be locally convex algebra endowed with a smooth action of G, of stabilization $\mathcal{A} = A \otimes_\pi \mathcal{D}$. The Dirac element α induces an isomorphism*

$$HP(\alpha \rtimes G) : HP_\bullet(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes G) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}(\mathcal{A} \rtimes G). \tag{4}$$

Combining these results yields the isomorphism (2), stated in theorem 4.20.

Periodic cyclic homology

Fix a complex complete locally convex algebra A. For $n \geq 0$, let us consider the vector space

$$C_n(A) := A^{\otimes_\pi n} \oplus A^{\otimes_\pi n+1}.$$

Its grading $C_\star(A) = \bigoplus_{n\geq 0} C_n(A)$ is endowed with the Hochschild's and Connes' differentials

$$b : C_n(A) \longrightarrow C_{n-1}(A) \quad \text{and} \quad B : C_n(A) \longrightarrow C_{n+1}(A) \tag{5}$$

verifying the property $b^2 = B^2 = bB + Bb = 0$, see [Lod92]. In particular, the sum $b + B$ is a differential of odd degree and we denote intuitively $C_{\text{even}}(A)$ and $C_{\text{odd}}(A)$ for the even and odd parts of $C_\star(A)$.

Definition 1.1. *The **periodic cyclic complex** of A is the following $\mathbb{Z}/2\mathbb{Z}$-graded complex and its homology the **periodic cyclic homology** of A:*

$$\widehat{CC}(A) := \; C_{\text{even}}(A) \underset{b+B}{\overset{b+B}{\rightleftarrows}} C_{\text{odd}}(A) \quad \textit{and} \quad HP_\bullet(A) := H_\bullet(\widehat{CC}(A)).$$

Theorem 1.2. *([Lod92] and [Pus98]) The periodic cyclic homology is stable under Morita equivalence, smooth homotopies and verifies excision.*

The periodic cyclic homology can be thought as an homological analogue of K-theory because there exists a natural and canonical *algebraic* Chern character

$$Ch : K_\bullet(A) \longrightarrow HP_\bullet(A)$$

from K-theory to periodic cyclic homology which extends the *geometrical* Chern character with De Rham cohomology as codomain.

Smooth crossed product algebras

Fix G to be real Lie group acting smoothly on a complete locally convex algebra and A.

Definition 1.3. *We defined the associated* ***smooth crossed product*** *as the convolution algebra*

$$A \rtimes G := \mathcal{C}_c^\infty(G,A) \quad (f_1 \star f_2)(g) = \int_G f_1(s)\ s\cdot(f_2(s^{-1}g))ds \in A.$$

It is endowed with a smooth action of G given by $(s\cdot f)(g) := s\cdot(f(s^{-1}g))$.

The aim of this paper is to understand the periodic cyclic homology of this algebra.

2 Morita equivalences of crossed product algebras

Consider as before a real reductive group G with maximal compact subgroup K, and a complete locally convex G-algebra A. In this section we establish the isomorphism (3)

$$HP_\bullet(A\rtimes K) \xrightarrow{\sim} HP_\bullet(\mathcal{C}_c^\infty(G/K,A)\rtimes G).$$

To do so, we will use the notion of *quasi-Morita equivalence* developed by J. Cuntz and D. Quillen [Cun98], [CQ95b] and [CQ95a] to obtain periodic cyclic complexes that are homotopy equivalent. This idea relies on the derived functor analogy of the periodic cyclic homology via quasi-free extensions. The datum of two G-algebras that are quasi-Morita equivalent yield to quasi-free extensions of the same algebra, which will canonically and naturally identifies the periodic cyclic complexes of their G-crossed product algebras. In the statement 2.10, we show that $\mathcal{C}_c^\infty(G/H,A)$ and $\mathcal{C}_c^\infty(G,A)\rtimes H$ are G-equivariantly quasi-Morita equivalent algebras for any closed subgroup $H<G$, which stands as a weak version of the Green-Imprimitivity theorem [Wil07] for Fréchet algebras. The periodic cyclic complexes of their crossed product will yield to the isomorphism above.

2.1 The Cuntz-Quillen method

In the 90-s, J. Cuntz and D. Quillen established a description of the periodic cyclic homology using quasi-free extension of algebras [CQ95a]. This point of view enriches the structure of periodic cyclic homology with a *derived functor property*.

We say that a locally convex algebra R is **quasi-free** if for any nilpotent linearly-split extension $0\longrightarrow I\longrightarrow S\longrightarrow R\longrightarrow 0$, there exists a morphism of algebras $f: R\longrightarrow S$. A **quasi-free extension** of a locally convex algebra A is a quasi-free algebra R which fits into a linearly-split extension of algebras

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] & R \arrow[r, "p"] & A \arrow[l, bend left, dashed, "s"] \arrow[r] & 0 \end{tikzcd}.$$

For instance, the **tensor algebra** TA associated to A is a free (hence quasi-free) algebra because any linear map $A\to B$ extends to a morphism of algebras $TA\to B$. Also, the multiplication ideal $IA=\ker(m: TA\to A)$ yields an exact sequence $0\longrightarrow IA\longrightarrow TA\xrightarrow{m} A\longrightarrow 0$ which makes TA a (quasi-)free extension of the algebra A.

Definition 2.1. *Define the* ***X-complex*** *of any complete locally convex algebra R as the $\mathbb{Z}/2\mathbb{Z}$-graded complex:*

$$X(R) := \begin{tikzcd} R \arrow[r, shift left, "B"] & C_1(R)/b(C_2(R)) \arrow[l, shift left, "b"] \end{tikzcd}.$$

where b and B denote the Hochschild's and Connes' differentials as in (5).

The motivation behind this complex comes from the fact that the X-complex $X(\widehat{TA})$ of the IA-adic completion of TA is chain-homotopy equivalent to the periodic cyclic complex $\widehat{CC}(A)$. It turns out that given any quasi-free extension $R = A/I$, the X-complex of the associated I-adic completion $X(\widehat{R})$ becomes chain-homotopy equivalent to $X(\widehat{TA})$. It endows the periodic cyclic homology with a *derived functor property* in the following sense.

Theorem 2.2 (Cuntz-Quillen [CQ95a]). *Let R be any quasi-free extension of A which fits into a linearly-split extension $0 \longrightarrow I \longrightarrow R \longrightarrow A \longrightarrow 0$. The X-complex associated to the I-adic completion $\widehat{R} := \varprojlim R/I^n$ is naturally and continuously chain-homotopy equivalent to the periodic cyclic complex of A*

$$X(\widehat{R}) \xrightarrow{\sim} \widehat{CC}(A).$$

Corollary 2.3. *Let V and W be two complex vector spaces which fit into an exact sequence of vector spaces $0 \longrightarrow V \xrightarrow{i} W \xrightarrow{p} A \longrightarrow 0$, with a linear section $s : A \to W$. There exists a canonical chain homotopy equivalence*

$$X(\widehat{TW}) \xrightarrow{\sim} \widehat{CC}(A)$$

where $\widehat{TW}$ is the completion of TW with respect to the ideal $I = \ker(m \circ Tp : TW \to A)$.

Proof. Since the tensor algebra TA is a quasi-free extension of the algebra A, the theorem above tells us that there exists a natural and continuous chain-homotopy equivalence $X(\widehat{TA}) \xrightarrow{\sim} \widehat{CC}(A)$ where $\widehat{TA}$ denotes the adic completion of the tensor algebra with respect to the ideal $IA = \ker(m : TA \to A)$. The statement will then follow from the fact that the X-complexes of $\widehat{TW}$ and $\widehat{TA}$ are chain-homotopy equivalent. To do so, we will show that the morphism of algebras $Tp : TW \to TA$ is an isomorphism of filtrated algebras, modulo homotopy.
Let us prove that $Ts : TA \to TW$ realizes an inverse modulo homotopy of $Tp : TW \to TA$. First, we show that Tp and Ts are continuous for the adic-topologies, namely that Tp and Ts are morphisms of filtrated algebras. If $x \in I \subseteq TW$ we get $m(Tp(x)) = (m \circ Tp)(x) = 0$, *i.e.* $Tp(x) \in IA$. On the other side, if $a \in IA \subseteq TA$, $(m \circ Tp)(Ts(a)) = (m \circ T(p \circ s))(a) = m(a) = 0$ which is that $Ts(a) \in I$.
Now, we show that Tp and Ts are inverse to each others modulo homotopy. By definition of s and p, it is clear that $Tp \circ Ts = id_{TA}$. Also, $(Ts \circ Tp)(Ti \star Ts) = T(s \circ p \circ i) \star T(s \circ p \circ s) = 0 \star Ts = (Ti \star Ts)(0 \star id)$ and the morphism $Ts \circ Tp$ fits into the following commutative diagram:

$$\begin{tikzcd}
TV \star TA \arrow[r, "Ti\star Ts"] \arrow[d, "0\star id"'] & TW \arrow[d, "Ts\circ Tp"] \\
TV \star TA \arrow[r, "Ti\star Ts"] & TW
\end{tikzcd}$$

Consider the family of linear maps $h_t : V \to V, v \mapsto tv$ for $t \in [0,1]$ and thus the morphisms $T(h_t) \star id : TV \star TA \to TV \star TA$. It is clear that the homotopy preserves the filtrations and that it is smooth with respect to the parameter $t \in [0,1]$. Since it realizes an homotopy between $0 \star id$ and the identity of $TV \star TA$, it realizes through $Ti \star Ts$ an homotopy between $Ts \circ Tp$ and the identity of TW. The fact that Tp is an isomorphism modulo homotopy would follow from the fact that the free homomorphism $Ti \star Ts$ is an isomorphism. Indeed, to show this free homomorphism is an isomorphism, we construct an inverse. The linear map $\phi : w \in W \longmapsto (w - (s \circ p)(w)) + p(w) \in TV \star TA$ extends to a morphism of algebras $id \star Tp : TW \longrightarrow TV \star TA$ by universal property of tensor algebras. Since $Tp \circ Ts = id$, it is clear that $(id \star Tp) \circ (Ti \star Ts) = id_{TV \star TA}$. On the other side

$$((Ti \star Ts) \circ \phi)(w) = (Ti \star Ts)((w - (s \circ p)(w)) + p(w)) = i(w - (s \circ p)(w)) + (s \circ p)(w) = w,$$

which implies that $(Ti \star Ts) \circ (id \star Tp) = id_{TW}$. Then $Ti \star Ts$ is a isomorphism and it ends up the proof. $\square$

2.2 Quasi-Morita equivalence

In this section, we propose a notion of *quasi-Morita equivalence* of algebras which is weaker than the notion of Morita equivalence, but still compatible with periodic cyclic homology.

Definition 2.4. *Two complete locally convex algebras A and B are* ***Morita equivalent*** *if there exists a (A, B)-bimodule E and a (B, A)-bimodule F such that the following morphisms of (A, A)-bimodules and (B, B)-bimodules are isomorphisms*

$$m_A : E \otimes_B F \longrightarrow A \quad \text{and} \quad m_B : F \otimes_A E \longrightarrow B.$$

The algebras A and B are said to be ***quasi-Morita equivalent*** *if the morphisms m_A and m_B are onto and linear split.*
We will use the notations $\sim_{\mathcal{M}}$ and $\sim_{q\mathcal{M}}$ for Morita and quasi-Morita equivalences.

Lemma 2.5. $A \sim_{\mathcal{M}} B \;\Rightarrow\; A \sim_{q\mathcal{M}} B$

Proof. It is clear that if m_A and m_B are isomorphisms of bimodules, they are onto and linear split. □

Proposition 2.6. *If A and B are two quasi-Morita equivalent complete locally convex algebras, then there exists a bounded chain homotopy equivalence*

$$\widehat{CC}(A) \xrightarrow{\sim} \widehat{CC}(B).$$

Proof. Suppose A and B are quasi-Morita equivalent. Then the morphisms of bimodules $m_A : E \otimes_B F \longrightarrow A$ and $m_B : F \otimes_A E \longrightarrow B$ are onto and linearly-split. The corollary 2.3 applies to the vector spaces $W = E \otimes_B F$ over A and $W = F \otimes_A E$ over B. We obtain two chain-homotopy equivalences

$$X\left(T(\widehat{E \otimes_B F})\right) \xrightarrow{\sim} \widehat{CC}(A) \quad \text{and} \quad X\left(T(\widehat{F \otimes_A E})\right) \xrightarrow{\sim} \widehat{CC}(B)$$

where we took the adic-completions with respect to the ideals $\ker(m_A)$ and $\ker(m_B)$. According to [CQ95a] and [Cun98], the linear map

$$\begin{array}{ccc} X(T(E \otimes_B F)) & \xrightarrow{\Theta} & X(T(F \otimes_A E)) \\ (e_0 \otimes f_0) \otimes (e_1 \otimes f_1) \otimes \cdots \otimes (e_n \otimes f_n) & \longmapsto & (f_n \otimes e_0) \otimes (f_0 \otimes e_1) \otimes \cdots \otimes (f_{n-1} \otimes e_n) \\ (e_0 \otimes f_0) \otimes \cdots \otimes (e_n \otimes f_n) d(e_{n+1} \otimes f_{n+1}) & \longmapsto & (f_{n+1} \otimes e_0) \otimes \cdots \otimes (f_{n+1} \otimes e_n) d(f_n \otimes e_{n+1}) \end{array} .$$

defines a continuous isomorphism for the adic and locally convex topologies. Since these X-complexes are dense in their respective adic completions, the assertion follows. □

Definition 2.7. *We say that A and B are* ***G-equivariantly quasi-Morita equivalent*** *if A and B are quasi-Morita equivalent G-algebras and that the associated bimodules E and F are endowed with a compatible action of G. We write $\sim_{G-q\mathcal{M}}$ for G-equivariant quasi-Morita equivalence.*

The notion of G-equivariant quasi-Morita equivalence enables us to compare not only their respective periodic cyclic complexes as in 2.6, but also the periodic cyclic complexes of their crossed product algebras. Initially established for the action of a discrete group by J. Cuntz [Cun98], the proof extends to the following framework.

Theorem 2.8. $A \sim_{G-q\mathcal{M}} B \;\Rightarrow\; A \rtimes G \sim_{q\mathcal{M}} B \rtimes G.$

Proof. Suppose that A and B are two G-equivariantly quasi-Morita equivalent complete locally convex algebras. By definition, there exists a (A,B)-bimodule E and a (B,A)-bimodule F, both endowed with a G-actions, such that the G-morphisms of (A,A)-bimodules and (B,B)-bimodules

$$m_A : E \otimes_B F \twoheadrightarrow A \quad \text{and} \quad m_B : F \otimes_A E \twoheadrightarrow B$$

are onto with linear sections s_A and s_B respectively. We will show that $A \rtimes G$ and $B \rtimes G$ are quasi-Morita equivalent.
Consider the vector spaces $E' = \mathcal{C}_c^\infty(G,E)$ and $F' = \mathcal{C}_c^\infty(G,F)$. It is clear that E' is a $(A \rtimes G, B \rtimes G)$-bimodule and F' is a $(B \rtimes G, A \rtimes G)$-bimodule for the actions by convolution. There exists some $(A \rtimes G, A \rtimes G)$-bimodules and $(B \rtimes G, B \rtimes G)$-bimodules isomorphisms

$$E' \otimes_{B \rtimes G} F' \xrightarrow{\sim} \mathcal{C}_c^\infty(G, E \otimes_B F), \quad \varphi_E \otimes \varphi_F \mapsto \left[g \mapsto \int_G \varphi_E(s) \otimes \varphi_F(s^{-1}g) ds \right],$$

$$F' \otimes_{A \rtimes G} E' \xrightarrow{\sim} \mathcal{C}_c^\infty(G, F \otimes_A E), \quad \varphi_F \otimes \varphi_E \mapsto \left[g \mapsto \int_G \varphi_F(s) \otimes \varphi_E(s^{-1}g) ds \right].$$

Since m_A and m_B are onto by assumption, the morphisms of $(A \rtimes G, A \rtimes G)$-bimodules and $(B \rtimes G, B \rtimes G)$-bimodules

$$m_{A \rtimes G} : \mathcal{C}_c^\infty(G, E \otimes_B F) \longrightarrow A \rtimes G \quad \text{and} \quad m_{B \rtimes G} : \mathcal{C}_c^\infty(G, F \otimes_A E) \longrightarrow B \rtimes G$$

are also onto. Finally, $m_{A \rtimes G}$ and $m_{B \rtimes G}$ are linearly-split via the sections given by

$$\begin{array}{cccc} s_{A \rtimes G} : & A \rtimes G & \longrightarrow & \mathcal{C}_c^\infty(G, E \otimes_B F) \\ & \varphi & \longmapsto & [g \mapsto s_A(\varphi(g))] \end{array} \quad \text{and} \quad \begin{array}{cccc} s_{B \rtimes G} : & B \rtimes G & \longrightarrow & \mathcal{C}_c^\infty(G, F \otimes_A E) \\ & \varphi & \longmapsto & [g \mapsto s_B(\varphi(g))] \end{array}.$$

It shows that $A \rtimes G$ and $B \rtimes G$ are quasi-Morita equivalent. □

Corollary 2.9. *If A and B are two G-equivariantly quasi-Morita equivalent complete locally convex algebras, then there exists a bounded chain homotopy equivalence*

$$\widehat{CC}(A \rtimes G) \xrightarrow{\sim} \widehat{CC}(B \rtimes G).$$

Proof. It is a direct consequence of the proposition 2.6 and the previous theorem. □

The previous statement means that the transformation $\widehat{CC}(- \rtimes G) : \boldsymbol{G\text{-}\mathbf{alg}} \longrightarrow \boldsymbol{\mathcal{H}\mathbf{o}(\mathcal{C})}$ viewed as a functor from the category of complete locally convex G-algebras to the chain-homotopy category is stable under G-equivariant quasi-Morita equivalences.

2.3 Green's Imprimitivity theorem

Let G be a real reductive group and fix a left-invariant Haar measure on it. Let also A be a unital Fréchet algebra endowed with a smooth action of G. Any closed subgroup $H < G$ acts on $\mathcal{C}_c^\infty(G,A)$ via the action

$$(h \cdot f)(g) := f(gh). \tag{6}$$

We built the crossed product $\mathcal{C}_c^\infty(G,A) \rtimes H$ with respect to this action.

Proposition 2.10. *For any unital algebra A endowed with a smooth action of G and any closed subgroup $H < G$, the following algebras are G-equivariantly quasi-Morita equivalent*

$$\mathcal{C}_c^\infty(G/H, A) \ \sim_{G-q\mathcal{M}} \ \mathcal{C}_c^\infty(G,A) \rtimes H$$

for the respective G-actions on $\mathcal{A} := \mathcal{C}_c^\infty(G/H, A)$ and $\mathcal{B} := \mathcal{C}_c^\infty(G,A) \rtimes H$ given by

$$(s \cdot a)(gH) = s \cdot a(s^{-1}gH) \quad \text{and} \quad (s \cdot b)(g,h) = s \cdot b(s^{-1}g, h). \tag{7}$$

Proof. Consider the vector G-spaces $E = F = \mathcal{C}_c^\infty(G, A)$ for the G-actions given by

$$(s \cdot e)(g) = s \cdot e(s^{-1}g) \text{ and } (s \cdot f)(g) = s \cdot f(s^{-1}g).$$

They are respectively equipped with a $(\mathcal{A}, \mathcal{B})$-bimodule and $(\mathcal{B}, \mathcal{A})$-bimodule structures given by

$$(a \cdot e \cdot b)(g) := \int_H a(gH)e(gh)b(gh, h^{-1})dh \text{ and } (b \cdot f \cdot a)(g) := \int_H b(g, h)f(gh)a(gH)dh.$$

Indeed, it is clear that the structures of right $\mathcal{A}$-module on E and left $\mathcal{A}$-module on F commute with the pointwise product on $\mathcal{A}$. We now verify that the structures of left $\mathcal{B}$-module on E and right $\mathcal{B}$-module on F commute with the convolution product on $\mathcal{B}$

$$\begin{aligned}
(e \cdot (b \star b'))(g) &= \int_H e(gh)(b \star b')(gh, h^{-1})dh \\
&= \int_H \int_H e(gh)b(gh, s)b'(ghs, s^{-1}h^{-1})dsdh \\
(hs \to s) &= \int_H \int_H e(gh)b(gh, h^{-1}s)b'(gs, s^{-1})dsdh \\
(h^{-1}s \to h^{-1}) &= \int_H \int_H e(gsh)b(gsh, h^{-1})b'(gs, s^{-1})dsdh \\
&= \int_H (e \cdot b)(gs)b'(gs, s^{-1})ds \\
&= ((e \cdot b) \cdot b')(g)
\end{aligned}$$

$$\begin{aligned}
((b \star b') \cdot f)(g) &= \int_H (b \star b')(g, h)f(gh)dh \\
&= \int_H \int_H b(g, s)b'(gs, s^{-1}h)f(gh)dsdh \\
(s^{-1}h \to h) &= \int_H \int_H b(g, s)b'(gs, h)f(gsh)dsdh \\
&= \int_H b(g, s)(b' \cdot f)(gs)ds \\
&= (b \cdot (b' \cdot f))(g)
\end{aligned}$$

These bimodule structures are also compatible with all the actions of G because we can check the following rules by direct computation

$$s \cdot (a \cdot e) = (s \cdot a) \cdot (s \cdot e),\ s \cdot (e \cdot b) = (s \cdot e) \cdot (s \cdot b),\ s \cdot (f \cdot a) = (s \cdot f) \cdot (s \cdot a),\ \text{and } s \cdot (b \cdot f) = (s \cdot b) \cdot (s \cdot f).$$

Define the two morphisms of A-bimodules an B-bimodules respectively

$$\begin{array}{cccc} m_{\mathcal{A}}: & E \otimes_\pi F & \longrightarrow & \mathcal{A} \\ & (e, f) & \longmapsto & \left[gH \mapsto \int_H e(gh)f(gh)dh\right] \end{array}$$

$$\begin{array}{cccc} m_{\mathcal{B}}: & F \otimes_\pi E & \longrightarrow & \mathcal{B} \\ & (f, e) & \longmapsto & [(g, h) \mapsto f(g)e(gh)] \end{array}.$$

It remains to find linear-sections of these morphisms.
Take a Bruhat function $\rho \in \mathcal{C}_c^\infty(G)$ associated to the bundle $\pi : G \longrightarrow G/H$, namely verifying that $\mathrm{supp}(\rho) \cap \pi^{-1}(K)$ is compact for any compact $K \subseteq G/H$ and $\int_H \rho(gh)dh = 1$ for any $g \in G$ [Bru61]. Consider the linear map

$$\begin{array}{cccc} s_{\mathcal{A}}: & \mathcal{A} & \longrightarrow & E \otimes_\pi F \\ & a & \longmapsto & [(g, h) \mapsto \rho(g)(1 \otimes a(gH))] \end{array}.$$

It is obviously smooth and its image is compactly-supported because $\mathrm{supp}(\rho) \cap \pi^{-1}(\mathrm{supp}(a))$ is compact as ρ is a Bruhat function. It defines a section of $m_{\mathcal{A}}$ as shows the following computation

$$(m_{\mathcal{A}} \circ s_{\mathcal{A}})(a)(gH) = \int_G \rho(gh)a(gH)dh = \left(\int_G \rho(gh)dh\right) a(gH) = a(gH).$$

Finally, the linear map

$$\begin{array}{cccc} s_{\mathcal{B}}: & \mathcal{B} & \longrightarrow & F \otimes_\pi E \\ & b & \longmapsto & \left[(g, h) \mapsto b(g, g^{-1}h)\right] \end{array}.$$

is a section of $m_{\mathcal{B}}$ following the computation $(m_{\mathcal{B}} \circ s_{\mathcal{B}})(b)(g, h) = s_{\mathcal{B}}(b)(g, gh) = b(g, h)$. We showed that $s_{\mathcal{A}}$ is a linear section of $m_{\mathcal{A}}$ and $s_{\mathcal{B}}$ is a linear section of $m_{\mathcal{B}}$, which gives the expected result since everything is compatible with the G-actions. ☐

Theorem 2.11 (Weak version of Green's imprimity theorem). *Let G be a real reductive group and $H < G$ a closed subgroup of it. For every unital algebra A endowed with a smooth action of G, there exists an isomorphism in periodic cyclic homology:*

$$HP_\bullet(A \rtimes H) \xrightarrow{\sim} HP_\bullet(\mathcal{C}_c^\infty(G/H, A) \rtimes G)$$

Applied to a maximal compact subgroup $K < G$, it realizes the isomorphism (3).

Proof. The previous proposition applied to G and the subgroup H yields the following equivariant quasi-Morita equivalences

$$A \ \sim_{H-q\mathcal{M}} \ \mathcal{C}_c^\infty(G, A) \rtimes G \quad \text{and} \quad \mathcal{C}_c^\infty(G/H, A) \sim_{G-q\mathcal{M}} \ \mathcal{C}_c^\infty(G, A) \rtimes H.$$

The first being a G-equivariant quasi-Morita equivalence it is in particular H-equivariant. Due to the corollary 2.9, it induces some chain-homotopy equivalences

$$\widehat{CC}(A \rtimes H) \xrightarrow{\sim} \widehat{CC}\Big((\mathcal{C}_c^\infty(G, A) \rtimes G) \rtimes H\Big) \ \text{and} \ \widehat{CC}(\mathcal{C}_c^\infty(G/H, A) \rtimes G) \xrightarrow{\sim} \widehat{CC}\Big((\mathcal{C}_c^\infty(G, A) \rtimes H) \rtimes G\Big).$$

We can rewrite these *double* crossed products as crossed product associated to $G \times H$

$$(\mathcal{C}_c^\infty(G, A) \rtimes G) \rtimes H \simeq \mathcal{C}_c^\infty(G, A) \rtimes_\alpha (G \times H) \ \text{ and } \ (\mathcal{C}_c^\infty(G, A) \rtimes H) \rtimes G \simeq \mathcal{C}_c^\infty(G, A) \rtimes_\beta (G \times H)$$

for the action α and β given by

$$\alpha_{(s,h)} f(g) = h \cdot f(h^{-1} g s) \ \text{ and } \ \beta_{(s,h)} f(g) := s \cdot f(s^{-1} g h)$$

by the definition (6) and (7). The proof follows from an identification of these actions under the linear map $\varphi : \mathcal{C}_c^\infty(G, A) \longrightarrow \mathcal{C}_c^\infty(G, A)$ defined by the formula $\varphi(f)(g) = g \cdot f(g^{-1})$. Indeed, it is an isomorphism of vector spaces which commutes with the pointwise multiplication on $\mathcal{C}_c^\infty(G, A)$ and transports the $G \times H$-action α to the $G \times H$-action β

$$\varphi(\alpha_{(s,h)} f)(g) = g \cdot \alpha_{(s,h)} f(g^{-1}) = gh \cdot f(h^{-1} g^{-1} s) = s \cdot \varphi(f)(s^{-1} g h) = \beta_{(s,h)} \varphi(f)(g).$$

The assertion follows. ∎

3 Dirac-dual Dirac method in KK-theory

Let G be a connected real reductive Lie group, K a maximal compact subgroup and A a complete locally convex algebra endowed with a smooth action of G. The aim of this section is to introduce all the theory needed to show the isomorphism (4), after stabilization

$$HP_\bullet(\mathcal{C}_c^\infty(G/K,A)\rtimes G) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}(A\rtimes G).$$

Lemma 3.1. *If the morphism above is an isomorphism for any pair (G,K) with G/K even-dimensional, then it is also an isomorphism for the pairs such that G/K is odd-dimensional.*

Proof. Suppose the morphism above is an isomorphism for any pair (G,K) with G/K even-dimensional. Take a pair (G,K) such that G/K is odd dimensional. Consider the group $G' = G\times\mathbb{R}$. The maximal compact subgroup of G' remains K and the quotient space is now $G/K\times\mathbb{R}$ of even dimension. The morphism above is then an isomorphism for the pair (G',K). We use the Connes-Thom isomorphism in periodic cyclic homology to get the degree shift. The assertion follows. $\square$

Assumption: In the whole section, the homogeneous space G/K **will be supposed even-dimensional** as a consequence of the previous lemma. In our case of interest, the isomorphism (4) becomes, after stabilization

$$HP_\bullet(\mathcal{C}_c^\infty(G/K,A)\rtimes G) \xrightarrow{\sim} HP_\bullet(A\rtimes G). \tag{8}$$

The idea to obtain this isomorphism (in the even-dimensional case) is to find a G-equivariant morphism of Fréchet algebras whose crossed product by G is a quasi-isomorphism in periodic cyclic homology. Indeed, the theorem 1 of V. Nistor [Nis93] establishes a condition to get a quasi-isomorphism after passing to crossed product.

Corollary 3.2. *Suppose $\varphi : A\to B$ is a morphism of G-algebras. Then*

$$HP(\varphi\rtimes K) \text{ is invertible } \Rightarrow HP(\varphi\rtimes G) \text{ is an isomorphism.}$$

Proof. The map φ extends into morphisms $\varphi\rtimes G : A\rtimes G\to B\rtimes G$ and $\varphi\rtimes K : A\rtimes K\to B\rtimes K$ as it is G-equivariant and in particular K-equivariant. The theorem 1 of Nistor tells us that for any maximal ideal $\mathfrak{m}$ of the smooth class functions algebra $\mathcal{C}^\infty(G)^{\mathrm{inv}}$ we have the following isomorphism, natural in A

$$HP_\bullet(A\rtimes G)_\mathfrak{m} \simeq HP_{\bullet+\dim(G/K)}(A\rtimes K)_\mathfrak{m}.$$

Using this theorem for the G-algebras A and B builds the following commutative diagram with vertical isomorphisms:

$$\begin{tikzcd}
HP_\bullet(A\rtimes G)_\mathfrak{m} \arrow[r, "HP(\varphi\rtimes G)"] \arrow[d, "\sim"'] & HP_\bullet(B\rtimes G)_\mathfrak{m} \arrow[d, "\sim"] \\
HP_{\bullet+\dim(G/K)}(A\rtimes K)_\mathfrak{m} \arrow[r, "HP(\varphi\rtimes K)"] & HP_{\bullet+\dim(G/K)}(B\rtimes K)_\mathfrak{m}
\end{tikzcd}$$

Indeed, as $HP(\phi\rtimes K)$ is supposed to be invertible, the bottom arrow becomes an isomorphism, and so is the the top arrow. We showed that $\varphi\rtimes G$ is globally defined and a quasi-isomorphism locally, and so is a global quasi-isomorphism. $\square$

Strategy 3.3. *Our strategy then relies on the existence of:*

- *a G-equivariant morphism α of Fréchet algebras inducing the linear map*

$$HP(\alpha \rtimes G) : HP_\bullet(\mathcal{C}_c^\infty(G/K, A) \rtimes G) \longrightarrow HP_\bullet(A \rtimes G);$$

- *a K-equivariant morphism β of Fréchet algebras such that $HP(\beta \rtimes K)$ is an inverse of $HP(\alpha \rtimes K)$.*

In that case, $HP(\alpha \rtimes G)$ would realize the expected isomorphism (8).

The morphisms α and β, which can be thought as K-equivariantly dual to each other, *implement Dirac and dual Dirac elements in periodic cyclic homology.* In this section we propose an introduction to the main results of the Dirac-dual Dirac construction in bivariant K-theory.

3.1 Dirac and dual Dirac elements

This section introduces the construction of the Dirac and dual Dirac elements and highlighting their behavior regarding to the Kasparov product.

Let q be an integer. The **Clifford algebra** of $\mathbb{R}^q$ is the *freest* unital associative algebra generated by $\mathbb{R}^q$ with product subject to the condition $x^2 = \|x\|^2 \cdot 1$, for all $x \in \mathbb{R}^q$. It is defined as

$$\mathrm{Cl}(q) := T\mathbb{R}^q / \langle x \otimes x - \|x\| \cdot 1,\ x \in \mathbb{R}^q \rangle.$$

If $e_1, \cdots, e_q$ is an orthogonal basis of $\mathbb{R}^q$, a basis of the Clifford algebra as a real vector space is given by $\{e_{i_1} \cdots e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq q \text{ and } 0 \leq k \leq q\}$. When q is even-dimensional, the Clifford algebra decomposes into some even-odd parts

$$\mathrm{Cl}(q) = \mathrm{Cl}(q)_+ \oplus \mathrm{Cl}(q)_- \tag{9}$$

generated by respectively even and odd elements of the basis. Let us use the notation $\mathrm{Cl}_{\mathbb{C}}(q) = \mathrm{Cl}(q) \otimes_{\mathbb{R}} \mathbb{C}$ for its complexification.

Recall that G is a real Lie group G and K a maximal compact subgroup of it. The homogeneous space G/K is a Euclidian space of dimension q even by assumption of the section §3. Let us call **Clifford module** and **smooth Clifford module** of G/K the following algebras

$$\mathcal{C}_\tau(G/K) = \mathcal{C}_0(G/K, \mathrm{Cl}_{\mathbb{C}}(q)) \ \text{ and } \ \mathcal{C}_\tau^\infty(G/K) = \mathcal{C}_c^\infty(G/K, \mathrm{Cl}_{\mathbb{C}}(q)).$$

These are algebras for the pointwise product $(s \cdot s')(x) = s(x) \cdot s'(x)$ where $\cdot$ denotes the Clifford multiplication. The first algebra is the $C^\star$-completion of the second one.

Theorem 3.4. *The canonical inclusion $i : \mathbb{C} \hookrightarrow \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q)$ induces a continuous morphism of algebras $i_\star : \mathcal{C}_c^\infty(G/K) \hookrightarrow \mathcal{C}_\tau^\infty(G/K)$ from smooth compactly-supported functions to the smooth Clifford module. For any Banach G-algebra A, this inclusion realizes a chain homotopy equivalence*

$$\widehat{CC}\Big(\mathcal{C}_c^\infty(G/K, A) \rtimes G\Big) \xrightarrow{\sim} \widehat{CC}\Big((\mathcal{C}_\tau^\infty(G/K) \otimes_\pi A) \rtimes G\Big).$$

Proof. Let us denote by $j : \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q) \longrightarrow \mathrm{End}(\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q))$ the morphism induced by the Clifford multiplication. We write $\widetilde{\mathrm{Tr}} = 2^{-q}\mathrm{Tr} : \mathrm{End}(\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q)) \longrightarrow \mathbb{C}$ for the normalized trace map, it verifies $\widetilde{\mathrm{Tr}}(1_{\mathrm{End}}) = 1$ because the dimension of $\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q)$ is exactly 2^q. The statement will follow from the proof that the composition $\widetilde{\mathrm{Tr}} \circ j : \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q) \longrightarrow \mathbb{C}$ realizes an inverse of the canonical inclusion, modulo homotopy, through the functor $\widehat{CC}((- \otimes_\pi A) \rtimes G)$. First, the composition:

$$\mathbb{C} \xrightarrow{i} \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q) \xrightarrow{j} \mathrm{End}(\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q)) \xrightarrow{\widetilde{\mathrm{Tr}}} \mathbb{C}$$

sends $1 \in \mathbb{C}$ to the normalize trace of the identity matrix 1_{End}, which is equal to $1 \in \mathbb{C}$. Therefore, the composition $\widetilde{\mathrm{Tr}}_\star \circ j_\star \circ i_\star$ is equal to the identity at the level of algebras, and becomes chain-homotopic to the identity through the functor $\widehat{CC}((- \otimes_\pi A) \rtimes G)$.
Now we show the other side of the composition. The following diagram commutes

$$\begin{tikzcd}
\Gamma_c^\infty\big(G/K, \mathrm{End}(\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q))\big) \arrow[r, "\widetilde{\mathrm{Tr}}_\star"] \arrow[d, "(id_{\mathrm{End}} \otimes i)_\star"'] & \mathcal{C}_c^\infty(G/K) \arrow[d, "i_\star"] \\
\Gamma_c^\infty\big(G/K, \mathrm{End}(\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q)) \otimes \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q)\big) \arrow[r, "(\widetilde{\mathrm{Tr}} \otimes id_{\mathrm{Cl}})_\star"] & \mathcal{C}_\tau^\infty(G/K)
\end{tikzcd}.$$

The composition $i_\star \circ \widetilde{\mathrm{Tr}}_\star \circ j_\star : \mathcal{C}_\tau^\infty(G/K) \longrightarrow \mathcal{C}_\tau^\infty(G/K)$ can then be computed as

$$(\widetilde{\mathrm{Tr}} \otimes id_{\mathrm{Cl}})_\star \circ (id_{\mathrm{End}} \otimes i)_\star \circ j_\star : \mathcal{C}_\tau^\infty(G/K) \longrightarrow \mathcal{C}_\tau^\infty(G/K).$$

For $0 \leq t \leq \pi/2$, the family of linear maps

$$\begin{array}{rccc} h_t : & \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q) & \longrightarrow & \mathrm{End}(\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q)) \otimes_\pi \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^q) \\ & v & \longmapsto & \cos(t)\big(j(v) \otimes 1_{\mathrm{Cl}}\big) + \sin(t)\big(1_{\mathrm{End}} \otimes v\big) \end{array}$$

realizes a G-equivariant smooth homotopy between $h_0 = (id_{\mathrm{End}} \otimes i) \circ j$ and $h_{\pi/2} = 1_{\mathrm{End}} \otimes id_{\mathrm{Cl}}$. Namely, it shows that $i_\star \circ \widetilde{\mathrm{Tr}}_\star \circ j_\star : \mathcal{C}_\tau^\infty(G/K) \longrightarrow \mathcal{C}_\tau^\infty(G/K)$ is G-equivariantly homotopic to the composition $(\widetilde{\mathrm{Tr}} \otimes id_{\mathrm{Cl}})_\star \circ (1_{\mathrm{End}} \otimes id_{\mathrm{Cl}})_\star = (\widetilde{\mathrm{Tr}}(1_{\mathrm{End}}) \otimes id_{\mathrm{Cl}})_\star = id_{\mathcal{C}_\tau^\infty(G/K)}$ because the normalized trace verifies $\widetilde{\mathrm{Tr}}(1_{\mathrm{End}}) = 1$. It makes the composition $i \circ \widetilde{\mathrm{Tr}} \circ j$ chain-homotopic to the identity at the level of periodic cyclic complexes through the functor $\widehat{CC}((- \otimes_\pi \mathcal{A}) \rtimes G)$, which ends up the proof. □

In the two next sections, we will define the Dirac and dual Dirac elements as classes in equivariant KK-groups associated to $\mathcal{C}_\tau(G/K)$.

3.1.1 The Dirac element

This paragraph provides a definition of the Dirac element as a G-equivariant Fredholm module over $\mathcal{C}_\tau(G/K)$, namely a class $\alpha \in KK_G(\mathcal{C}_\tau(G/K), \mathbb{C})$. We recall here the analytic model for equivariant K-homology from the references [Val02] and [HR01].

Definition 3.5. *A G-**equivariant Fredholm module** over a complete locally convex G-algebra A is a triplet (ρ, F, U) where:*

- *$\rho : A \to \mathcal{L}(H)$ is a bounded representation;*
- *F a bounded self-adjoint operator on H;*
- *$U : G \to U(H)$ is a unitary representation;*

which verify for any $g \in G$ and $a \in A$ that

$$\rho(a)(F^2 - 1) \in \mathcal{K}(H), \quad \rho(a)F - F\rho(a) \in \mathcal{K}(H), \quad U_g F = F U_g \quad \textit{and} \quad U_g \rho(a) U_g^{-1} = \rho(g \cdot a). \tag{10}$$

A G-equivariant Fredholm module (ρ, F, U) is **even** if the underlying Hilbert space decomposes as $H = H_+ \oplus H_-$, ρ and U are degree preserving and F is odd. Otherwise, the Fredholm module is said to be **odd**. Two G-equivariant Fredholm modules (ρ, F, U) and (ρ', F', U') are **operator homotopic** if $\rho = \rho'$, $U = U'$ and if there exists a norm continuous path $(F_t)_{t\in[0,1]}$ connecting F to F' such that for each $t \in [0,1]$ and $g \in G$, $U_g F_t = F_t U_g$.

Definition 3.6. *We define the group $KK_G(A, \mathbb{C})$ to be the abelian group of operator homotopy classes of even G-equivariant Fredholm modules over A.*

Let us fix $\mathcal{H} = L^2(\Omega^\star(G/K))$ to be the L^2-completion of the compactly supported differential forms $\Omega_c^\star(G/K)$ over the homogeneous space G/K. Under the identification of the Clifford algebra $\mathrm{Cl}(q)$ with the exterior algebra $\bigwedge \mathbb{R}^q$ it is clear that $\mathcal{H}$ is a module over the Clifford module $\mathcal{C}_\tau(G/K)$. Namely, the Clifford multiplication induces a bounded representation

$$\begin{array}{rccc} \rho: & \mathcal{C}_\tau(G/K) & \longrightarrow & \mathcal{L}(\mathcal{H}) \\ & s & \longmapsto & f \mapsto [x \mapsto s(x)\cdot f(x)] \end{array}.$$

Consider also the De-Rham Laplace operator $D := d + d^\star : \mathcal{H} \longrightarrow \mathcal{H}$. Its *bounded version* is a self-adjoint bounded operator on $\mathcal{H}$

$$F := D(1 + D^2)^{-1/2} \in \mathcal{L}(\mathcal{H}).$$

Finally, the real Lie group G acts by left-translation on the homogeneous space G/K and preserves the metric. It induces a representation of the group on $\mathcal{H}$ given by:

$$\begin{array}{rccc} U: & G & \longrightarrow & \mathcal{L}(\mathcal{H}) \\ & g & \longmapsto & f \mapsto [x \mapsto f(g\cdot x)] \end{array}.$$

Theorem 3.7 (Kasparov, [Kas88]). *The triplet $\alpha := (\rho, F, U)$ is an even G-equivariant Fredholm module over $\mathcal{C}_\tau(G/K)$, hence represents a class $\alpha \in KK_G(\mathcal{C}_\tau(G/K), \mathbb{C})$ that we call **Dirac element**.*

3.1.2 The dual Dirac element

Now we propose a definition the dual Dirac element as a class in the K-equivariant K-theory of $\mathcal{C}_\tau(G/K)$, which is a class $\beta \in KK_K(\mathbb{C}, \mathcal{C}_\tau(G/K))$. We recall a model for equivariant K-theory.

Fix a complete locally convex algebra A endowed with an action of the maximal compact subgroup K. We say that two matrices $p \in M_n(A)$ and $q \in M_m(A)$ are **stable-equivalent** if there exist an integer $r \geq \max(m, n)$ and a matrix $u \in GL_r(A)$ such that:

$$u \begin{pmatrix} p & 0 \\ 0 & 0_{r-n} \end{pmatrix} u^{-1} = \begin{pmatrix} q & 0 \\ 0 & 0_{r-m} \end{pmatrix}.$$

The set of stable-equivalent classes of K-equivariant idempotent matrices over A is a monoid for the direct sum.

Definition 3.8. *We define the group $KK_K(\mathbb{C}, A)$ to be the Grothendieck group of stable-equivalent classes of K-equivariant idempotent matrices over A.*

Consider a smooth function $\chi : \mathbb{R}^q \to \mathbb{R}$ invariant under rotations which is identically 1 on the ball $B(0, 1/2)$ and identically zero out of $B(0,1)$. We define two smooth maps

$$e_0, e_1 : \mathbb{R}^q \longrightarrow M_2(\mathrm{Cl}(q))$$

which associate to any point $x \in \mathbb{R}^q$ the idempotent matrices

$$e_0(x) = \frac{1}{\chi(x)^2 + \|x\|^2} \begin{pmatrix} \chi(x)^2 & \chi(x)\cdot x \\ \chi(x)\cdot x & \|x\|^2 \end{pmatrix} \quad \text{and} \quad e_1(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorem 3.9. *The difference of these smooth functions defines a class* $\beta = [e_0] - [e_1] \in KK_K(\mathbb{C}, \mathcal{C}_\tau(G/K))$ *that we call* ***dual Dirac element****.*

Proof. For any $x \in \mathbb{R}^q$, $e_0(x)$ and $e_1(x)$ are idempotent matrices over $\mathrm{Cl}_\mathbb{C}(q)$ so e_0 and e_1 are idempotent matrices over $\mathcal{C}^\infty(\mathbb{R}^q, \mathrm{Cl}(q))$. When x moves away from the origin, $e_0(x) = e_1(x)$ which makes their difference $e_0 - e_1$ a matrix with compactly supported coefficients, *i.e.* an element of $M_2(\mathcal{C}^\infty_c(\mathbb{R}^q, \mathrm{Cl}_\mathbb{C}(q)))$. The homeomorphism between the homogeneous space G/K and the Euclidian space $\mathbb{R}^q$ transforms the translation K-action on G/K into the rotation-action on $\mathbb{R}^q$. So it is clear that under this homeomorphism both e_0 and e_1 commutes with the action of K and that $e_0 - e_1 \in M_2(\mathcal{C}^\infty_\tau(G/K))$ is a K-equivariant difference of idempotent matrices. Since the $C^\star$-completion of $\mathcal{C}^\infty_\tau(G/K)$ is $\mathcal{C}_\tau(G/K)$, the difference $e_0 - e_1$ represents a class in $KK_K(\mathbb{C}, \mathcal{C}_\tau(G/K))$ as expected. □

Theorem 3.10 (Kasparov [Kas81])**.** *The Kasparov product of the dual Dirac element* β *with the Dirac element* α *is trivial in both direction:*

$$\alpha \times \beta = 1 \in KK_G(\mathcal{C}_\tau(G/K), \mathcal{C}_\tau(G/K)) \quad \textit{and} \quad Res^G_K(\beta \times \alpha) = 1 \in KK_K(\mathbb{C}, \mathbb{C}).$$

The Dirac and dual Dirac elements can be thought as K-equivariant dual to each others. These elements will realize the K-equivariant dual morphisms of Fréchet algebras α and β we want to obtain in our strategy 3.3. The next section proposes a way to realize these bivariant K-theory classes as $\star$-homomorphisms.

3.2 Cuntz' approach to KK-theory

We would like to get algebraic description of the Dirac and dual Dirac elements in order to adapt their construction to periodic cyclic homology. This algebraic description is provided by the Cuntz-picture which identifies bivariant theory groups $KK(A, B)$ as homotopy classes of $\star$-homomorphisms from the Cuntz' algebra qA to $B \otimes_{C^\star} \mathcal{K}$, where $\mathcal{K}$ denote the $C^\star$-algebra of compact operators on a infinite dimensional Hilbert space. In this setting, the Kasparov product corresponds to the composition of the underlying homomorphisms.

Cuntz' algebra

Let A and B be two complex algebras. The **algebraic free product** $A \star B$ is the algebra generated by words with letters alternating between A and B, modulo the relations internal to each algebra:

$$A \star B := A \oplus B \oplus (A \otimes B) \oplus (B \otimes A) \oplus (A \otimes B \otimes A) \oplus (B \otimes A \otimes B) \oplus \cdots$$

It is naturally equipped with two canonical inclusions $\iota_0 : A \to A \star B$ and $\iota_1 : B \to A \star B$. When $A = B$, we define the **free product** QA to be the universal ring generated by the two copies $\iota_0(A)$ and $\iota_1(A)$. It verifies the semi-universal property that any pair of homomorphisms $\phi, \psi : A \to B$ induces a homomorphism $\phi \star \psi : QA \to B$ such that $(\phi \star \psi) \circ \iota_0 = \phi$ and $(\phi \star \psi) \circ \iota_1 = \psi$.

Definition 3.11. *The* ***Cuntz's algebra*** qA *associated to an algebra* A *is the kernel of the fold map* $id \star id : QA \to A$ *arising from the pair of identity morphisms*

$$qA := \ker(id \star id : QA \to A).$$

It fits into the following doubly-split extension of algebras

$$\begin{tikzcd} 0 \arrow[r] & qA \arrow[r, hook] & QA \arrow[r, "id\star id"] & A \arrow[l, dashed, bend left, "\iota_0,\ \iota_1"] \arrow[r] & 0 \end{tikzcd}.$$

If the algebra A is endowed with a $C^\star$-structure, then its free product QA is also a $C^\star$-algebra for the largest possible $C^\star$-norm, namely:

$$\|x\|_{QA} := \{\|\pi(x)\| \mid \pi : QA \to \mathcal{L}(H) \text{ is a } \star-\text{representation into a Hilbert space } H\}.$$

The Cuntz's algebra qA being a subspace of it, it inherits the induced topology. It verifies the following fundamental property.

Theorem 3.12 (Cuntz, [Cun87]). *Up to stabilization by square matrices there exists an homotopy equivalence $qA \xrightarrow{\sim} q(qA)$ for any separable $C^\star$-algebra A.*

Cuntz' identification

The Cuntz' approach was motivated by the search of an algebraic description of the bivariant K-theory. Indeed, K-theory and K-homology groups should both encode differences of $\star$-homomorphisms, and the kernel qA of the fold map is precisely the universal $C^\star$-algebra in which such differences become genuine algebraic elements. It is the universal receptacle of difference of $\star$-homomorphisms out of A. This identifies qA as the natural object representing bivariant K-theory.

Theorem 3.13 (Cuntz, [Cun87]). *Let G be a locally compact group and (A, B) a pair of separable $C^\star$-algebras endowed with a continuous action of G. Any G-equivariant KK-theory class uniquely determines a continuous G-equivariant $\star$-homomorphism modulo homotopy*

$$KK_G(A, B) \hookrightarrow \mathrm{Hom}_G(qA, B \otimes_{C^\star} \mathcal{K})_{/\sim}.$$

If the group is compact, this description is an one-to-one correspondence.

This approach provides a simple description of the equivariant Kasparov product. Given A, B and C three $C^\star$-algebras endowed with a continuous action of a compact group K we can *compose* some continuous K-equivariant maps $f : qA \to B \otimes_{C^\star} \mathcal{K}$ and $g : qB \to C \otimes_{C^\star} \mathcal{K}$ in the following way

$$g \circ qf : qA \xrightarrow[3.12]{\sim} q(qA) \longrightarrow q(B \otimes_{C^\star} \mathcal{K}) \longrightarrow C \otimes_{C^\star} \mathcal{K} \otimes_{C^\star} \mathcal{K} \simeq C \otimes_{C^\star} \mathcal{K}. \tag{11}$$

This composition defines a class in $KK_K(A, C)$ which depends only on the homotopy classes of f and g.

Theorem 3.14 (Cuntz, [Cun87]). *Given two classes $[f] \in KK_K(A, B)$ and $[g] \in KK_K(B, C)$, there exists an homotopy equivalence between their Kasparov product and their composition in the sense of (11). Namely, the followings define the same class in bivariant K-theory:*

$$[f] \times [g] = [g \circ qf] \in KK_K(A, C).$$

Composition of Dirac and dual Dirac elements

Consider a Hilbert space $\mathcal{H}$ endowed with a unitary representation $\pi : G \to \mathcal{L}(\mathcal{H})$. We define the **$G$-smooth p^{th}-Schatten ideal** of $\mathcal{H}$ as the ideal:

$$\ell^{p,G}(\mathcal{H}) := \left\{ T \in \mathcal{L}(\mathcal{H}) \;\middle|\; \begin{array}{l} T \in \ell^p(\mathcal{H}), \text{ namely } \mathrm{Tr}((T^\star T)^p) < \infty \\ g \mapsto \pi(g) T \pi(g)^{-1} \text{ is infinitely differentiable for all } g \in G \end{array} \right\}. \tag{12}$$

It is the greatest subalgebra of the p^{th}-Schatten ideal $\ell^p(\mathcal{H})$ which is endowed with a smooth action of G.

Lemma 3.15. *(Kasparov [Kas81]) Let P be a projection from $\mathcal{H}$ onto a one dimensional subspace of K-fixed vectors. The trivial class $1 = [\mathbb{1}_K] \in KK_K(\mathbb{C},\mathbb{C})$ can be represented by the composition of algebra homomorphisms*

$$\begin{array}{rcccl} \mathbb{1}_K : q(q\mathbb{C}) & \xrightarrow{(id\star 0)\circ(id\star 0)} & \mathbb{C} & \xrightarrow{\quad\iota\quad} & M_2(\ell^{p,G}(\mathcal{H})) \\ & & 1 & \longmapsto & \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \end{array} .$$

Recall that the Dirac element $\alpha \in KK_G(\mathcal{C}_\tau(G/K),\mathbb{C})$ and the dual Dirac element $\beta \in KK_K(\mathbb{C},\mathcal{C}_\tau(G/K))$ define equivariant KK-classes. The theorem 3.13 shows that they can be represented as a G-equivariant and a K-equivariant $\star$-homomorphisms

$$\alpha : q(\mathcal{C}_\tau(G/K)) \longrightarrow \mathcal{K} \quad \text{and} \quad \beta : q\mathbb{C} \longrightarrow M_2(\mathcal{C}_\tau(G/K)).$$

Indeed, the image of the dual Dirac element lies in the algebra of 2×2 matrices since it is defined using matrices of this size (3.9).

Corollary 3.16. *The following morphisms of algebras are K-equivariantly homotopy equivalent to each others*

$$\alpha \circ q\beta : q(q\mathbb{C}) \longrightarrow M_2(\mathcal{K}) \quad \textit{and} \quad \mathbb{1}_K : q(q\mathbb{C}) \longrightarrow M_2(\ell^{p,G}(\mathcal{H})) \subseteq M_2(\mathcal{K}).$$

Proof. By the Kasparov theorem 3.10 we know that the K-restriction of their Kasparov product $\beta \times \alpha$ is trivial. Then, the theorem 3.14 implies that their composition $\alpha \circ q\beta$ is K-equivariantly homotopy equivalent to a representative of the trivial class in K-equivariant KK-theory. The assertion follows from the lemma abvove. $\square$

4 Dirac-dual Dirac method in periodic cyclic homology

The aim of this section is to show that the Dirac element $\alpha \in KK_G(\mathcal{C}_\tau(G/K),\mathbb{C})$ realizes the expected isomorphism (8) of periodic cyclic homology groups

$$HP((\alpha \otimes id_{\mathcal{A}}) \rtimes G) : HP_\bullet(\mathcal{C}_c^\infty(G/K,\mathcal{A}) \rtimes G) \xrightarrow{\sim} HP_\bullet(\mathcal{A} \rtimes G)$$

after stabilization $\mathcal{A} = A \otimes_\pi \mathcal{D}$ with a certain Schatten ideal $\mathcal{D}$. To obtain this result, we will *descend* the construction of the previous section from bivariant K-theory to periodic cyclic homology by refining the Cuntz identification 3.13 for the Dirac and dual Dirac elements. We will show that we can associate to them Fréchet algebra homomorphisms that are K-equivariantly inverses to each other at the level of periodic cyclic complexes, as expected in the strategy 3.3.

4.1 Rescaled version of the Cuntz algebra

We first adapt the Cuntz' algebra qA defined in the previous section into a rescaling family of algebras $\{q_R A\}_{R>0}$ playing an analogous role for bounded homomorphisms of Fréchet algebras instead of $C^\star$-homomorphisms.

The rescaled free product

Let A be a Banach algebra. In order to work in a Banach framework, it is necessary to equip the algebraic free product $QA = A \star A$ with a submultiplicative norm compatible with given norm on the topological algebra A. Given $R \geq 1$, we define the **R-norm** $\|\cdot\|_R$ to be the largest norm which coincide on each factor $A^{\otimes_\pi n}$ with nR^{n-1} times the projective crossed norm. This sub-multiplicative norm may be thought as a rescaled version of the projective norm.

Definition 4.1. *Given a Banach algebra A and $R \geq 1$, we define the* **rescaled R-free product** *$Q_R A$ to be the completion of the algebraic free product $A \star A$ with respect to the norm $\|\cdot\|_R$.*

The algebra $Q_R A$ is a rescaled version of the classical $C^\star$-free product QA and allows to control a larger class of homomorphisms than just the contractive ones, as shows the next lemmas.

Lemma 4.2. *For any pair of Banach algebra homomorphisms $\phi_0, \phi_1 : A \to B$ of norm at most R there exists a unique homomorphism $\phi_0 \star \phi_1 : Q_R A \to B$ verifying:*

$$(\phi_0 \star \phi_1) \circ \iota_0 = \phi_0, \ (\phi_0 \star \phi_1) \circ \iota_1 = \phi_1 \text{ and } \|\phi_0 \star \phi_1\| \leq R.$$

It is given by $(\phi_0 \star \phi_1)(a_0 \cdots a_n) = \phi_0(a_0)\phi_1(a_1)\phi_0(a_2) \cdots \phi_{n \bmod 2}(a_n)$ on the reduced words.

Proof. It is clear that the definition of $\phi_0 \star \phi_1$ on the reduced words extends to a well-defined homomorphism on the whole algebra $Q_R A$. One can easily check the computations $(\phi_0 \star \phi_1) \circ \iota_0 = \phi_0$ and $(\phi_0 \star \phi_1) \circ \iota_1 = \phi_1$. It remains to show that $\phi_0 \star \phi_1$ is of norm at most R with respect to $\|\cdot\|_R$. Taking $x = \sum_{i \in I} a_0^{(i)} \cdots a_{n_i}^{(i)} \in A \star A$ we compute:

$$\begin{aligned}
\|(\phi_0 \star \phi_1)(x)\|_B &\leq \sum_{i \in I} \|(\phi_0 \star \phi_1)(a_0^{(i)} \cdots a_{n_i}^{(i)})\|_B. \\
&= \sum_{i \in I} \|\phi_0(a_0^{(i)})\|_B \|\phi_1(a_1^{(i)})\|_B \cdots \|\phi_{n_i \bmod 2}(a_{n_i}^{(i)})\|_B \\
&\leq \sum_{i \in I} R^{n_i+1} \|a_0^{(i)}\|_A \cdots \|a_{n_i}^{(i)}\|_A && \text{(because } \|\phi_0\|, \|\phi_1\| \leq R) \\
&\leq R \left(\sum_{i \in I} (n_i + 1) R^{n_i} \|a_0^{(i)}\|_A \cdots \|a_{n_i}^{(i)}\|_A \right) && \text{(because } n_i \geq 1) \\
&\leq R \|x\|_R.
\end{aligned}$$

It ends up the proof. □

Lemma 4.3. *Any bounded algebra morphism $\phi : A \to B$ induces a morphism of between the respective free algebras $Q\phi : Q_{R'} A \longrightarrow Q_R B$ when $R' \geq R\|\phi\|$.*

Proof. Composing ϕ with the canonical inclusions $\iota_0, \iota_1 : B \to Q_R B$ leads to a pair of algebra homomorphisms $(\iota_0 \circ \phi), (\iota_1 \circ \phi) : A \longrightarrow Q_R B$. It induces an algebra homomorphism $Q_{R'} A \to Q_R B$ for all R' greater to the norm of both $\iota_0 \circ \phi$ and $\iota_1 \circ \phi$. We compute $\|(\iota_0 \circ \phi)(a)\|_R = \|\phi(a)\|_R \leq R\|\phi\| \times \|a\|_A$. Same thing occurs for $\iota_1 \circ \phi$, which gives the expected result. □

The above lemma shows that the construction $A \rightsquigarrow Q_R A$ is functorial with respect to bounded algebra homomorphisms, provided adjustment of the parameter R. This flexibility is crucial for our setup.

The two inclusions on first and second coordinates $A \rightrightarrows A \oplus A$ are isometries. By definition of the free product, they induce for all $R \geq 1$ an algebra homomorphism we denote $i : Q_R A \to A \oplus A$ which verifies for all $a \in A$ the identities $i(\iota_0(a)) = (a, 0)$ and $i(\iota_1(a)) = (0, a)$. The following theorem is proved in [Cun87] for the $C^\star$-algebraic setting and we adapted the proof for this framework.

Theorem 4.4. *For all $R \geq 1$, the algebra homomorphism $i : Q_R A \to A \oplus A$ is homotopic to the identity modulo stabilization by 2×2-square matrices. Moreover, if A is a G-algebra, the homotopy equivalence is G-equivariant.*

Remark Before to show the theorem, we need to fix a topology on the space of square matrices. If B is a Banach algebra, we endow $M_2(B)$ with the topology induced by the multiplicative Hilbert-Schmidt norm. With this norm the space of square matrices with coefficients in B is Banach and the matrices

$$u_t := \begin{pmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix} \tag{13}$$

are isometries for all $t \in [0,1]$.

Proof. Let us take the algebra homomorphism

$$\begin{array}{cccc} j: & A \oplus A & \longrightarrow & M_2(Q_R A) \\ & (a,b) & \longmapsto & \begin{pmatrix} \iota_0(a) & 0 \\ 0 & \iota_1(b) \end{pmatrix} \end{array}.$$

We will prove that it defines an inverse to i modulo smooth homotopy. First, let us consider the composition $M_2(i) \circ j : A \oplus A \to M_2(A \oplus A)$. We want to show that it is homotopic to the top-left corner inclusion of $A \oplus A$ in its space of square matrices. We compute for $(a,b) \in A \oplus A$:

$$(M_2(i) \circ j)(a,b) = \begin{pmatrix} i(\iota_0(a)) & 0 \\ 0 & i(\iota_1(b)) \end{pmatrix} = \begin{pmatrix} (a,0) & 0 \\ 0 & (0,b) \end{pmatrix}.$$

The continuous family of homomorphisms

$$\begin{array}{cccc} w(t): & A \oplus A & \longrightarrow & M_2(A \oplus A) \\ & (a,b) & \longmapsto & \begin{pmatrix} (a,0) & 0 \\ 0 & 0 \end{pmatrix} u_t \begin{pmatrix} 0 & 0 \\ 0 & (0,b) \end{pmatrix} u_t^{-1} \end{array}$$

where u_t is the rotation matrix (13) realizes a smooth homotopy between $w(0) = M_2(i) \circ j$ and the top-left corner inclusion $w(1) = A \oplus A \hookrightarrow M_2(A \oplus A)$.
Now consider the other side composition $j \circ i : Q_R A \to M_2(Q_R A)$. We want to check that it is smoothly homotopic to the top-left corner inclusion of $Q_R A$ in its space of square matrices. For all $a \in A$, the morphism $j \circ i$ verifies, by definition of i as a free product, the following identities:

$$(j \circ i)(\iota_0(a)) = j(a,0) = \begin{pmatrix} \iota_0(a) & 0 \\ 0 & 0 \end{pmatrix} \text{ and } (j \circ i)(\iota_1(a)) = j(a,0) = \begin{pmatrix} 0 & 0 \\ 0 & \iota_1(a) \end{pmatrix}.$$

Since u_t is an isometry for the Hilbert-Schmidt norm for all $t \in [0,1]$, the following pair of isometric homomorphisms

$$\begin{array}{cccc} & A & \longrightarrow & M_2(Q_R A) \\ \gamma_0 : & a & \longmapsto & \begin{pmatrix} \iota_0(a) & 0 \\ 0 & 0 \end{pmatrix} \\ \gamma_1(t) : & a & \longmapsto & u_t \begin{pmatrix} 0 & 0 \\ 0 & \iota_1(a) \end{pmatrix} u_t^{-1} \end{array}$$

induces for all $R \geq 1$ an algebra homomorphism $\gamma_0 \star \gamma_1(t) : Q_R A \to M_2(Q_R A)$. When $t = 0$ we recover exactly $j \circ i$ by the computation above and when $t = 1$ we get the top-left corner inclusion $Q_R A \hookrightarrow M_2(Q_R A)$. We proved that the algebra homomorphism i is smoothly homotopic to identity modulo square matrices, which ends up the proof. $\square$

The rescaled Cuntz' algebra

Definition 4.5. *For all $R \geq 1$, the* ***rescaled Cuntz's algebra*** $q_R A$ *associated to a Banach algebra A is the kernel of the fold map $id \star id : Q_R A \to A$ arising from the pair of identity morphisms*

$$q_R A := \ker(id \star id : Q_R A \to A).$$

It fits into the following doubly-split extension of algebras

$$0 \longrightarrow q_R A \hookrightarrow Q_R A \underset{\iota_0,\ \iota_1}{\overset{id\star id}{\rightleftarrows}} A \longrightarrow 0 \ .$$

The next lemma shows that the rescaled Cuntz' algebra $q_R A$ is the *completion of the two-sided ideal of differences* in $Q_R A$. Indeed, it shows that the reduced words of $q_R A$ belongs in the two-sided ideal of differences in $Q_R A$ while any difference $\iota_0(a) - \iota_1(a)$ with $a \in A$ belongs to $q_R A$ by the universal property of the free product.

Lemma 4.6. *Any reduced word of the form $x = \iota_0(a_0)\iota_1(a_1)\cdots\iota_{n \text{ mod } 2}(a_n) \in q_R A$ with $a_i \in A$ can be written as:*

$$x = \sum_{k=0}^{\lfloor n/2 \rfloor} \iota_1(a_0 \cdots a_{2k-1})\Big(\iota_0(a_{2k}) - \iota_1(a_{2k})\Big)\iota_1(a_{2k+1})\iota_0(a_{2k+1})\cdots\iota_{n \text{ mod } 2}(a_n).$$

Proof. The telescopic sum above is an element of this ideal and a straightforward computation shows that it is equal to $x - \iota_1(a_0 \cdots a_n) \in Q_R A$. But because x is in the kernel of the fold map,

$$0 = (id \star id)(x) = (id \star id)(\iota_0(a_0)\iota_1(a_1)\cdots\iota_{n \bmod 2}(a_n)) = a_0 \cdots a_n \in A.$$

Then the sum above is exactly equal to x and we showed the lemma. □

The following theorem is central in the study. It relies on the fact that the homotopy equivalence between the free product $Q_R A$ and $A \oplus A$ is enough stable to be preserved under smooth actions of group and tensorization. Again, it is a rescaled and equivariant version of the constructions of [Cun87].

Theorem 4.7. *If A and $\mathcal{A}$ are G-algebras, the algebra homomorphism $id \star 0 : q_R A \to A$ induces a chain-homotopy equivalence through the functor $\widehat{CC}((- \otimes_\pi \mathcal{A}) \rtimes G)$, which is*

$$\widehat{CC}\big(((id \star 0) \otimes_\pi id_{\mathcal{A}}) \rtimes G\big) : \widehat{CC}\big((q_R A \otimes_\pi \mathcal{A}) \rtimes G\big) \xrightarrow{\sim} \widehat{CC}\big((A \otimes_\pi \mathcal{A}) \rtimes G\big).$$

Proof. Tensoring the double-split extension $0 \to q_R A \to Q_R A \to A \to 0$ with $\mathcal{A}$, and then taking the G-crossed product yields the extension

$$0 \longrightarrow (q_R A \otimes_\pi \mathcal{A}) \rtimes G \longrightarrow (Q_R A \otimes_\pi \mathcal{A}) \rtimes G \longrightarrow (A \otimes_\pi \mathcal{A}) \rtimes G \longrightarrow 0$$

which is doubly-split via the linear maps $(\iota_0 \otimes id_{\mathcal{A}}) \rtimes G$ and $(\iota_1 \otimes id_{\mathcal{A}}) \rtimes G$. The argument of Cuntz 4.4 showing that $Q_R A \to A \oplus A$ is a stable homotopy equivalence also applies to $(Q_R A \otimes_\pi \mathcal{A}) \rtimes G \longrightarrow (A \otimes_\pi \mathcal{A}) \rtimes G \oplus (A \otimes_\pi \mathcal{A}) \rtimes G$, which implies the expected result. □

Corollary 4.8. *For any G-algebras A, A', $\mathcal{A}$ and $R > 0$, the morphisms:*

$$\Delta_R = \Big(\big(\iota_0^A \otimes id\big) \star \big(\iota_1^A \otimes id\big) \Big)_{|q_R} : q_R(A \otimes_\pi A') \longrightarrow q_R A \otimes_\pi A',$$

$$\nabla_R = \Big(\big(id \otimes \iota_0^{A'}\big) \star \big(id \otimes \iota_1^{A'}\big) \Big)_{|q_R} : q_R(A \otimes_\pi A') \longrightarrow A \otimes_\pi q_R A'$$

induce chain-homotopy equivalences trough the functor $\widehat{CC}((- \otimes_\pi \mathcal{A}) \rtimes G)$.

Proof. It is a consequence of the identities

$$((id_A \star 0) \otimes id_{A'}) \circ \Delta_R = id_{A\otimes_\pi A'} \star 0 \quad \text{and} \quad (id_A \otimes (id_{A'} \star 0)) \circ \nabla_R = id_{A\otimes_\pi A'} \star 0$$

and the fact that $id \star 0$ is a chain-homotopy equivalences through the functor $\widehat{CC}((- \otimes_\pi \mathcal{A}) \rtimes G)$ due to the theorem above. □

Remark Recall that the Riemannian connection ∇ acts as a derivation on the smooth Clifford module $\mathcal{C}^\infty_\tau(G/K)$. We endow this algebra with Fréchet topology coming from the family of $\mathcal{C}^k$-sup norms $\|f\|_k := \|\nabla^k(f)\|$ with $k \in \mathbb{N}$ controlling the growth of the derivatives. Hence, for any $R > 0$, the associated Cuntz' algebra $q_R(\mathcal{C}^\infty_\tau(G/K))$ is equipped with a family of norms $\|\cdot\|_{(R,m)}$ with $m \in \mathbb{N}$ defined as

$$\|\iota_1(f_1)\cdots\iota_{n \bmod 2}(f_n)\|_{(R,m)} := \sum_{\substack{(m_1,\cdots,m_n)\in\mathbb{N}^n \\ m_1+\cdots m_n \le m}} R^n \|f_1\|_{m_1}\cdots\|f_n\|_{m_n}. \tag{14}$$

We fix this Fréchet topology on $q_R(\mathcal{C}^\infty_\tau(G/K))$ for the rest of the paper.

4.2 Fréchet version of Dirac and dual Dirac elements

In this subsection, we verify that there exists a parameter $R > 0$ great enough so that the Dirac and dual Dirac elements descend to respectively G-equivariant and K-equivariant morphisms of Fréchet algebras

$$\alpha^\sharp : q_R(\mathcal{C}^\infty_\tau(G/K)) \longrightarrow \ell^{p,G}(\mathcal{H}) \quad \text{and} \quad \beta^\sharp : q_1\mathbb{C} \longrightarrow \mathcal{C}^\infty_\tau(G/K) \otimes_\pi M_2(\mathbb{C})$$

where $\ell^{p,G}(\mathcal{H})$ is the G-smooth p-th Schatten ideal (definition 12) of a certain infinite dimensional Hilbert space $\mathcal{H}$. We will also compute their composition $\alpha^\sharp \circ q\beta^\sharp$ which turns out to be K-equivariantly homotopic to the morphism $\mathbb{1}_K$ representing the trivial class $1 \in KK_K(\mathbb{C},\mathbb{C})$ (see 3.15), as in the $C^\star$-algebraic framework.

Proposition 4.9. *The Dirac element $\alpha \in KK_G(\mathcal{C}_\tau(G/K),\mathbb{C})$ can be represented by a G-equivariant Kasparov bimodule (ρ, F, U) such that for $p > \dim(G/K)$ there exists a constant $R > 0$ verifying for all $f \in \mathcal{C}_\tau(G/K)$:*

$$F^2 = 1, \quad \|\rho(f)\| \le R\|f\|_0 \quad \textit{and} \quad \|[F,\rho(f)]\| \le R\|f\|_1.$$

Proof. According to Connes [Con85, pages 89-91] one may obtain, after adding a degenerate Kasparov bimodule to α, a G-invariant Fredholm module (ρ, F, U) which represents the same class as α in the G-equivariant KK-group $KK_G(C_\tau(G/K),\mathbb{C})$, is p-summable over $\mathcal{C}^\infty_c(G/K)$ for $p > dim(G/K)$ and satisfies $F^2 = Id$. It is clear that the morphism ρ is bounded because the Clifford multiplication representing α was already bounded. We have to show that the commutators $[F,\rho(f)]$ is also for the norm ℓ^p, when $p > \dim(G/K)$. Let $x \in G/K$ and $B_1 \subset B_2 \subset B_3 \subset G/K$ be balls of center x and strictly increasing radii. We will show that $[F,\rho(f)]$ is ℓ^p-bounded for any $f \in \mathcal{C}^\infty_c(B_1)$. Fix a real cutoff function $\chi \in \mathcal{C}^\infty_c(B_2)$ which is constant equal to 1 on B_1 and $\mu \in \mathcal{C}^\infty_c(B_3)$ which is constant equal to 1 on B_2. The commutator becomes:

$$[F,\rho(f)] = [F,\rho(f)]\chi + \rho(f)[F,\chi].$$

The image of the operator $\rho(f)[F,\chi]$ consists of functions supported in B_1. Viewing them as functions on $S^n \approx G/K \cup \{\infty\}$ vanishing outside B_1 we may interprete $\rho(f)[F,\chi]$ as pseudodifferential operator from G/K to S^n. Its order is -1 so that it extends to a bounded operator

$\rho(f)[F,\chi] : \mathcal{L}^2(G/K) \longrightarrow \mathcal{H}^{(1)}(S^n)$ whose norm is majorized in terms of f, χ and their first derivatives, i.e. in terms of $\|f\|_1$. Now we may factorize

$$\rho(f)[F,\chi] : \mathcal{L}^2(G/K) \longrightarrow \mathcal{H}^{(1)}(S^n) \longrightarrow \mathcal{L}^2(S^n) \xrightarrow{\cdot\mu} \mathcal{L}^2(G/K)$$

where the last map is given by multiplication with μ, followed by extension by 0 outside B_3. The first and last maps are bounded operators whereas the middle one lies in the p-th Schatten ideal for p great enough. The assertion for $\rho(f)[F,\chi]$ follows because ℓ^p is a two-sided ideal in $\mathcal{L}(H)$. Consider now the bounded operator $[F,\rho(f)]\chi$ on $\mathcal{L}^2(G/K)$. The operator F is bounded and self-adjoint on $\mathcal{L}^2(G/K)$. Now a bounded operator on Hilbert space belongs to the Schatten ideal ℓ^p if and only if its adjoint does and the Schatten norms of the two operators coincide. We find

$$([F,\rho(f)]\chi)^\star = -\chi[F,\rho(f)]$$

and our previous arguments apply to the latter operator. This proves that $[F,\rho(f)]$ is bounded for the ℓ^p-norm. Choose R great enough with respect to the norm of ρ and of the commutator $[F,\rho(f)]$. □

Since G/K is supposed to be even-dimensional, write the underlying Hilbert space as a direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and denote P_+ for the projection to the even part. Consider the morphisms

$$\rho_0 := P_+\rho P_+ : \mathcal{C}^\infty_\tau(G/K) \longrightarrow \mathcal{L}(\mathcal{H}) \quad \text{and} \quad \rho_1 := P_+F\rho(f)FP_+ : \mathcal{C}^\infty_\tau(G/K) \longrightarrow \mathcal{L}(\mathcal{H}).$$

Theorem 4.10. *The Dirac element α induces, for $p > \dim(G/K)$ and $R > 0$ great enough, a bounded G-equivariant morphism of Fréchet algebras*

$$\alpha^\sharp = (\rho_0 \star \rho_1)_{|q_R} : q_R(\mathcal{C}^\infty_\tau(G/K)) \longrightarrow \ell^{p,G}(\mathcal{H}).$$

where we recall that $\ell^{p,G}(\mathcal{H})$ is the G-smooth p-th Schatten ideal (definition 12).

Proof. Take R as in the lemma above. Due to the topology in $q_R(\mathcal{C}^\infty_\tau(G/K))$ coming from the $\mathcal{C}^k$-sup norms on $\mathcal{C}^\infty_\tau(G/K)$, it suffices to show that for any element $x = \iota_0(f_0)\iota_1(f_1)\cdots\iota_{n \bmod 2}(f_n) \in q_R(\mathcal{C}^\infty_\tau(G/K))$, it exists a constant $C > 0$ and an integer $m \in \mathbb{N}$ such that

$$\|(\rho_0 \star \rho_1)(x)\|_{\ell^p(H)} \le C\|x\|_{(R,m)}$$

where the norm $\|\cdot\|_{(R,m)}$ have been defined in (14). Due to the lemma 4.6, we can write

$$(\rho_0 \star \rho_1)(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \rho_1(f_0\cdots f_{2k-1})\Big(\rho_0(f_{2k}) - \rho_1(f_{2k})\Big)\rho_1(f_{2k+1})\cdots\rho_{n \bmod 2}(f_n).$$

The free product $\rho_0 \star \rho_1$ applied to $x \in q_RA$ is of norm:

$$\begin{aligned}
\|(\rho_0 \star \rho_0)(x)\|_{\ell^p(H)} &\le \sum_{k=0}^{\lfloor n/2 \rfloor} \|\rho_1(f_0\cdots f_{2k-1})\Big(\rho_0(f_{2k}) - \rho_1(f_{2k})\Big)\rho_1(f_{2k+1})\cdots\rho_{n \bmod 2}(f_n)\|_{\ell^p(H)} \\
&\le \sum_{k=0}^{\lfloor n/2 \rfloor} \|\rho_1(f_0\cdots f_{2k-1})\|_{L(H)}\|\rho_0(f_{2k}) - \rho_1(f_{2k})\|_{\ell^p(H)}\|\rho_1(f_{2k+1})\cdots\rho_{n \bmod 2}(f_n)\|_{L(H)} \\
&\le \sum_{k=0}^{\lfloor n/2 \rfloor} R^n\|f_0\|_0\cdots\|f_{2k-1}\|_0\|\rho_0(f_{2k}) - \rho_1(f_{2k})\|_{\ell^p(H)}\|f_{2k+1}\|_0\cdots\|f_n\|_0 && \text{(by 4.9)} \\
&\le \sum_{k=0}^{\lfloor n/2 \rfloor} R^{n+1}\|f_0\|_0\cdots\|f_{2k-1}\|_0\|f_{2k}\|_1\|f_{2k+1}\|_0\cdots\|f_n\|_0 && \text{(by 4.9)} \\
&\le R\|x\|_{(R,1)},
\end{aligned}$$

which ends up the proof of boundness. Finally, the image of $\rho_0 \star \rho_1$ is contained in the space of smooth vectors of the G-module $\ell^p(\mathcal{H})$ because F is G-equivariant and acts smoothly on $\mathcal{C}^\infty_\tau(G/K)$. ☐

Proposition 4.11. *The dual Dirac element $\beta = [e_0] - [e_1] \in KK_K(\mathbb{C}, \mathcal{C}_\tau(G/K))$ induces a K-equivariant bounded morphism of Fréchet algebras*

$$\beta^\sharp = (e_0 \star e_1)_{|q_1\mathbb{C}} : q_1\mathbb{C} \longrightarrow M_2(\mathcal{C}^\infty_\tau(G/K)).$$

Proof. It is a consequence of the universal property 4.2 of the Banach free algebras. The idempotent 2×2-matrices e_0 and e_1 are both of norm 1 and their difference is compactly-supported. ☐

Proposition 4.12. *The following algebra homomorphisms are K-equivariantly homotopy equivalent*

$$\alpha^\sharp \circ q_R\beta^\sharp : q_R(q_1\mathbb{C}) \longrightarrow M_2(\ell^{p,G}(\mathcal{H})) \quad \textit{and} \quad \mathbb{1}_K : q_R(q_1\mathbb{C}) \longrightarrow M_2(\ell^{p,G}(\mathcal{H})).$$

Proof. This assertion is a consequence of the corollary 3.16. In the proof of Kasparov [Kas88], the constructions occur at the level of the dense Fréchet subalgebras $\mathcal{C}^\infty_\tau(G/K) \subseteq \mathcal{C}_\tau(G/K)$ and $\ell^{p,G}(\mathcal{H}) \subseteq \mathcal{K}$ and then extends to the $C^\star$-completion. ☐

Remark We expected these versions of the Dirac and dual Dirac elements in periodic cyclic homology to be K-equivariantly inverse to each others, as for the Dirac and dual Dirac elements in bivariant K-theory. Unfortunately, the last proposition states that one side of their composition is not the identity morphism but a representative of the trivial class in $KK_K(\mathbb{C},\mathbb{C})$. Indeed, we don't know if the morphism $\mathbb{1}_K$ realizes the identity morphism in periodic cyclic homology because we are unable to compute the periodic cyclic homology of its codomain $\ell^{p,G}(\mathcal{H})$. We are able to prove that the p-th Schatten ideal $\ell^p(\mathcal{H})$ is quasi-Morita equivalent to $\ell^1(\mathcal{H})$ for any $p > 1$ and that $\ell^1(\mathcal{H})$ possesses trivial periodic cyclic homology but we still don't know anything about its subalgebra of G-smooth vectors.

In the next section we will define a stabilization algebra $\mathcal{D}$ so that when we take the tensor product by $\mathcal{D}$, the algebra morphism $\mathbb{1}_K$ becomes the identity. This algebra contains the degeneracy of the Dirac and dual Dirac in periodic cyclic homology to be inverse to each others. This is the reason why our main result is established *after stabilization*.

4.3 The stabilization algebra

Consider the infinite dimensional Hilbert space $\mathcal{H} = L^2(\Omega^\star(G/K)) \oplus \mathbb{C}$, the L^2-completion of the compactly supported differential forms $\Omega^\star_c(G/K)$, with an extra copy of $\mathbb{C}$. It is endowed with the G-action by translations on G/K and the trivial action on $\mathbb{C}$. The rank-one projection $P : \mathcal{H} \oplus \mathbb{C} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}$ is isometric and G-smooth, namely $P \in \ell^{p,G}(\mathcal{H})$ for any integer p. Since its image is made of K-fixed vectors the lemma 3.15 states that the following morphism represents the class of $1 \in KK_K(\mathbb{C},\mathbb{C})$:

$$\begin{array}{rcccl} \mathbb{1}_K : q(q\mathbb{C}) & \xrightarrow{(id\star 0)\circ(id\star 0)} & \mathbb{C} & \xrightarrow{\quad\iota\quad} & M_2(\ell^{p,G}(\mathcal{H})) \\ & & 1 & \longmapsto & \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \end{array}.$$

Consider the system of inclusions of algebras $\ell^{p,G}(\mathcal{H})^{\otimes_\pi n} \hookrightarrow \ell^{p,G}(\mathcal{H})^{\otimes_\pi n+1}$, $1 \longmapsto P \otimes 1$. The projection P being isometric, the system is compatible with the successive norms.

Definition 4.13. *The **stabilization algebra** $\mathcal{D}$ is defined to be the Banach completion of the inductive limit of the system above, for the inductive norm*

$$\mathcal{D} := \overline{\lim_{\to n} \ell^{p,G}(\mathcal{H})^{\otimes_\pi n}}.$$

*Given any Banach G-algebra A, we will write $\mathcal{A} := A \otimes_\pi \mathcal{D}$ for its **stabilized algebra**. Since $\mathcal{D}$ is endowed with a smooth action of G, the stabilization $\mathcal{A}$ is again a Banach G-algebra.*

Lemma 4.14. *For every G-algebra A with stabilization $\mathcal{A} = A \otimes_\pi \mathcal{D}$, the following chain-complex homomorphism is an chain-homotopy equivalence*

$$(\iota \otimes id_{\mathcal{A}}) \rtimes G : \widehat{CC}(\mathcal{A} \rtimes G) \xrightarrow{\sim} \widehat{CC}\Big(\big(M_2(\ell^{p,G}(\mathcal{H})) \otimes_\pi \mathcal{A}\big) \rtimes G \Big).$$

Proof. The stabilization algebra $\mathcal{D}$ is naturally isomorphic to $\ell^{p,G}(\mathcal{H}) \otimes_\pi \mathcal{D}$ by definition of the inductive limit and under this identification, the shift morphism $1 \in \mathcal{D} \longmapsto P \otimes 1 \in \ell^{p,G}(\mathcal{H}) \otimes_\pi \mathcal{D}$ is *equal* to the identity of $\mathcal{D}$. In particular, after tensoring with A, we also get the identification $\ell^{p,G}(\mathcal{H}) \otimes_\pi \mathcal{A} \simeq \mathcal{A}$ and that $(1 \mapsto P) \otimes id_{\mathcal{A}}$ is equal to $id_{\mathcal{A}}$. Then, the morphism

$$\iota \otimes id_{\mathcal{A}} : \mathcal{A} \longrightarrow M_2(\ell^{p,G}(\mathcal{H})) \otimes_\pi \mathcal{A} \simeq M_2(\mathcal{A})$$

is nothing other than the top-left inclusion of $\mathcal{A}$ in its 2×2-matrices algebra. Everything is compatible with the action of G as P is G-smooth. □

The next corollary shows that the stabilization algebra plays the expected role. Namely, after tensorization by this algebra, the Dirac element $\alpha^\sharp$ realizes the linear map (8) and the dual Dirac element $\beta^\sharp$ provides a K-equivariant right-inverse of it.

Corollary 4.15. *For any Banach G-algebra A with stabilization algebra $\mathcal{A} = A \otimes_\pi \mathcal{D}$,*

1. *the G-equivariant morphism of Fréchet algebras $\alpha^\sharp$ induces a linear map*

$$HP((\alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes G) : HP_\bullet(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes G) \longrightarrow HP_\bullet(\mathcal{A} \rtimes G);$$

2. *the K-equivariant morphism of Fréchet algebras $\beta^\sharp$ is such that $HP((\beta^\sharp \otimes id_{\mathcal{A}}) \rtimes K)$ realizes a **right-inverse** of $HP((\alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes K)$.*

Proof. 1. Recall that the Dirac element α induces a G-equivariant morphism of Fréchet algebras $\alpha^\sharp : q_R(\mathcal{C}_\tau^\infty(G/K)) \longrightarrow M_2(\ell^{p,G}(\mathcal{H}))$ by theorem 4.10. According to the theorems 3.4 and 4.7 the we have the following chain-homotopy equivalences

$$\widehat{CC}\Big((q_R(\mathcal{C}_\tau^\infty(G/K)) \otimes_\pi \mathcal{A}) \rtimes G \Big) \xrightarrow[4.7]{\sim} \widehat{CC}\Big((\mathcal{C}_\tau^\infty(G/K) \otimes_\pi \mathcal{A}) \rtimes G \Big) \xleftarrow[3.4]{\sim} \widehat{CC}(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes G).$$

Moreover, the previous lemma establishes a chain-homotopy equivalence

$$\widehat{CC}(\mathcal{A} \rtimes G) \xrightarrow[4.14]{\sim} \widehat{CC}\Big(\big(M_2(\ell^{p,G}(\mathcal{H})) \otimes_\pi \mathcal{A}\big) \rtimes G \Big).$$

Combining both of these descriptions yields to the first assertion.
2. The proposition 4.12 states that there exists a K-equivariant homotopy equivalence between the morphisms of algebras

$$\alpha^\sharp \circ q_R\beta^\sharp : q_R(q_1\mathbb{C}) \longrightarrow M_2(\ell^{p,G}(\mathcal{H})) \quad \text{and} \quad \mathbb{1}_K = \iota \circ (id \star 0)^2 : q_R(q_1\mathbb{C}) \longrightarrow M_2(\ell^{p,G}(\mathcal{H})).$$

The second homomorphism is chain-homotopy equivalent to the identity through the functor $\widehat{CC}((- \otimes_\pi \mathcal{A}) \rtimes K)$ because both $id \star 0$ and ι are also by the theorem 4.7 and the lemma 4.14 above. The linear map $HP((q_R\beta^\sharp \otimes id_{\mathcal{A}}) \rtimes K)$ realizes then a right-inverse of $HP((\alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes K)$. Finally, $HP((q_R\beta^\sharp \otimes id_{\mathcal{A}}) \rtimes K) = HP((\beta^\sharp \otimes id_{\mathcal{A}}) \rtimes K)$ by 4.7. □

4.4 Rotation trick

In the previous section we established that the dual Dirac element provides a K-equivariant left-inverse of the Dirac element, so we are now interested in the right-invertibility of the Dirac element. To do so, we will use an analogue of Atiyah's rotation trick to compute a side of their composition using their other-side composition. At the bivariant K-theory level, this has to be thought as the computation of the Kasparov product $\alpha \times \beta$ using the other-side product $\beta \times \alpha$. We will realize this rotation trick using the flip of coordinates in the Euclidian manifold $G/K \times G/K$. **For convenience, we will write $\mathcal{M}$ for the algebra** $M_2(\mathcal{C}^\infty_\tau(G/K))$.

Lemma 4.16 (Rotation trick)**.** *Recall the definition of the dual Dirac element β using the idempotent matrices $e_0, e_1 \in \mathcal{M}$ (3.9). The rotation that flips the coordinates in $G/K \times G/K$ realizes a K-equivariant smooth homotopy between the following morphisms, for $i = 0, 1$:*

$$e_i \otimes id : \mathcal{M} \longrightarrow \mathcal{M} \otimes_\pi \mathcal{M} \quad \textit{and} \quad id \otimes e_i : \mathcal{M} \longrightarrow \mathcal{M} \otimes_\pi \mathcal{M}.$$

Proof. Since by assumption G/K is even dimensional, the smooth Clifford module splits into even and odd parts $\mathcal{C}^\infty_\tau(G/K) = \mathcal{C}^\infty_\tau(G/K)_+ \oplus \mathcal{C}^\infty_\tau(G/K)_-$ as in the equation (9). We write ε for the grading automorphism which is the identity on its even part and minus the identity on its odd part. A direct computation shows that for any $x \in \mathbb{R}^q$ and $i = 0, 1$ we have $\varepsilon(e_i(-x)) = e_i(x)$. Namely, it suffices to show that the following algebra homomorphism is smoothly homotopic to the identity:

$$\begin{array}{cccc} \Theta: & \mathcal{C}^\infty_\tau(G/K) \otimes_\pi \mathcal{C}^\infty_\tau(G/K) & \longrightarrow & \mathcal{C}^\infty_\tau(G/K) \otimes_\pi \mathcal{C}^\infty_\tau(G/K) \\ & a \otimes b & \longmapsto & (x,y) \mapsto b(x) \otimes \varepsilon(a(-y)) \end{array}.$$

Recall the identification

$$\mathcal{C}^\infty_\tau(G/K) \otimes_\pi \mathcal{C}^\infty_\tau(G/K) \simeq \mathcal{C}^\infty_c(G/K \times G/K, \mathrm{Cl}_\mathbb{C}(\mathbb{R}^q \oplus \mathbb{R}^q)).$$

The smooth family of rotations for $0 \leq t \leq \pi/2$

$$h_t : G/K \times G/K \longrightarrow G/K \times G/K, \ (v, v') \mapsto (\cos(t)v + \sin(t)v', -\sin(t)v + \cos(t)v')$$

joins the map $(x, y) \mapsto (-y, x)$ to the identity and the smooth family of linear isometries for $0 \leq t \leq \pi/2$

$$h'_t : \mathbb{R}^q \oplus \mathbb{R}^q \longrightarrow \mathbb{R}^q \oplus \mathbb{R}^q, \ (v, v') \mapsto (\cos(t)v + \sin(t)v', -\sin(t)v + \cos(t)v')$$

induce a smooth homotopy of algebra automorphisms $\varphi_t : \mathrm{Cl}_\mathbb{C}(\mathbb{R}^q \oplus \mathbb{R}^q) \longrightarrow \mathrm{Cl}_\mathbb{C}(\mathbb{R}^q \oplus \mathbb{R}^q)$ between $\varphi_0 = id$ and $\varphi_{\pi/2}(a \otimes b) = b \otimes \varepsilon(a)$. The composition $\varphi_t \circ \Theta \circ h_t$, $0 \leq t \leq \frac{\pi}{2}$ provides the expected smooth homotopy. It is K-equivariant because the homeomorphism $G/K \sim \mathbb{R}^q$ transports the K-action to the action by rotations on $\mathbb{R}^q$. $\square$

Proposition 4.17. *The following algebra morphisms are chain-homotopy equivalent through the functor $\widehat{CC}((- \otimes A) \rtimes K)$ for any K-algebra A:*

$$\beta^\sharp \otimes id_\mathcal{M} : q_1\mathbb{C} \otimes_\pi \mathcal{M} \longrightarrow \mathcal{M} \otimes_\pi \mathcal{M} \quad \textit{and} \quad id_\mathcal{M} \otimes \beta^\sharp : \mathcal{M} \otimes_\pi q_1\mathbb{C} \longrightarrow \mathcal{M} \otimes_\pi \mathcal{M}$$

Proof. By definition of Δ and ∇ in 4.8, it is clear that the following diagrams commute

$$\begin{tikzcd} q_1\mathcal{M} \arrow[r, "(e_0 \otimes id) \star (e_1 \otimes id)"] \arrow[d, "\Delta_1"'] & \mathcal{M} \otimes_\pi \mathcal{M} \arrow[d, equal] \\ q_1\mathbb{C} \otimes_\pi \mathcal{M} \arrow[r, "\beta^\sharp \otimes id_\mathcal{M}"] & \mathcal{M} \otimes_\pi \mathcal{M} \end{tikzcd} \qquad \begin{tikzcd} q_1\mathcal{M} \arrow[r, "(id \otimes e_0) \star (id \otimes e_1)"] \arrow[d, "\nabla_1"'] & \mathcal{M} \otimes_\pi \mathcal{M} \arrow[d, equal] \\ \mathcal{M} \otimes_\pi q_1\mathbb{C} \arrow[r, "id_\mathcal{M} \otimes \beta^\sharp"] & \mathcal{M} \otimes_\pi \mathcal{M} \end{tikzcd}$$

The previous lemma states that the top arrows are K-equivariantly homotopy equivalent via the coordinate flip. The vertical maps are chain-homotopy equivalent to the identity through the functor $\widehat{CC}((- \otimes A) \rtimes K)$ via corollary 4.8. It ends up the proof. $\square$

Proposition 4.18. *For any Banach G-algebra A with stabilization algebra $\mathcal{A} = A \otimes_\pi \mathcal{D}$, the linear map $HP((\alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes K)$ is **left-invertible**.*

Proof. Since the coordinates are independant to each others, the following diagram commutes

$$\begin{tikzcd}
q_1\mathbb{C} \otimes_\pi q_R\mathcal{M} \arrow[r, "id_{q_1}\otimes\alpha^\sharp"] \arrow[d, "\beta^\sharp\otimes id_{q_R\mathcal{M}}"'] & q_1\mathbb{C} \otimes_\pi M_2(\ell^{p,G}(\mathcal{H})) \arrow[d, "\beta^\sharp\otimes id_{M_2}"] \\
\mathcal{M} \otimes_\pi q_R\mathcal{M} \arrow[r, "id_{\mathcal{M}}\otimes\alpha^\sharp"'] & \mathcal{M} \otimes_\pi M_2(\ell^{p,G}(\mathcal{H}))
\end{tikzcd}$$

The theorem 4.7 states that the left-vertical arrow is nothing other than $\beta^\sharp \otimes id_{\mathcal{M}}$ through the functor $\widehat{CC}((- \otimes \mathcal{A}) \rtimes K)$. But through this functor, $\beta^\sharp \otimes id_{\mathcal{M}}$ is chain-homotopy equivalent to $id_{\mathcal{M}} \otimes \beta^\sharp$ by the previous proposition. It shows that

$$(\beta^\sharp \otimes id_{M_2}) \circ (id_{q_1} \otimes \alpha^\sharp) = (id_{\mathcal{M}} \otimes \alpha^\sharp) \circ (\beta^\sharp \otimes id_{q_R\mathcal{M}}) \sim_{\widehat{CC}((-\otimes\mathcal{A})\rtimes K)} (id_{\mathcal{M}} \otimes \alpha^\sharp) \circ (id_{\mathcal{M}} \otimes \beta^\sharp).$$

Moreover, the second point of corollary 4.15 states that the last composition is chain-homotopy equivalent to the identity through the functor $\widehat{CC}((- \otimes \mathcal{A}) \rtimes K)$. Namely, $\widehat{CC}((\beta^\sharp \otimes id_{M_2} \otimes id_{\mathcal{A}}) \rtimes K)$ realizes a left-inverse of $\widehat{CC}((id_{q_1} \otimes \alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes K)$, which is chain-homotopy equivalent to $\widehat{CC}((\alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes K)$ by theorem 4.7. □

Corollary 4.19. *Let G be a reductive group and K be a maximal compact subgroup of it. For every complete locally convex G-algebra A, the Dirac element $\alpha \in KK_G(\mathcal{C}_\tau(G/K), \mathbb{C})$ induces an isomorphism in periodic cyclic homology*

$$(\alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes G : HP_\bullet(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes G) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}(\mathcal{A} \rtimes G)$$

where $\mathcal{A} = A \otimes_\pi \mathcal{D}$ is the stabilized algebra.

Proof. When G/K is even-dimensional, the corollary 4.15 and the proposition 4.18 together realize the hypothesis of the strategy 3.3 which makes isomorphism occur in this case. The odd-dimensional case being a consequence of the lemma 3.1. □

Theorem 4.20 (Main theorem)**.** *Let G be a reductive group and K be a maximal compact subgroup. For every complete locally convex G-algebra A, the Dirac element α induces an isomorphism in periodic cyclic homology*

$$HP((\alpha^\sharp \otimes id_{\mathcal{A}}) \rtimes G) : HP_\bullet(\mathcal{A} \rtimes K) \simeq HP_\bullet(\mathcal{C}_c^\infty(G/K, \mathcal{A}) \rtimes G) \xrightarrow{\sim} HP_{\bullet+\dim(G/K)}(\mathcal{A} \rtimes G)$$

where $\mathcal{A} = A \otimes_\pi \mathcal{D}$ is the stabilized algebra.

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